J. Math. Anal. Appl. 457 (2018) 616–644

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Continuous approximation of linear impulsive systems and a new form of robust stability

Kevin E.M. Church [a], Robert Smith? [b,*]

[a] Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada
[b] Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada

ARTICLE INFO

Article history:
Received 24 August 2016
Available online 24 August 2017
Submitted by J. Shi

Keywords:
Impulsive differential equations
Impulse extension
Stability
Robust stability
Time-scale tolerance
Exponential regulator

ABSTRACT

The time-scale tolerance for linear ordinary impulsive differential equations is introduced. How large the time-scale tolerance is directly reflects the degree to which the qualitative dynamics of the linear impulsive system can be affected by replacing the impulse effect with a continuous (as opposed to discontinuous, impulsive) perturbation, producing what is known as an impulse extension equation. Theoretical properties related to the existence of the time-scale tolerance are given for periodic systems, as are algorithms to compute them. Some methods are presented for general, aperiodic systems. Additionally, sufficient conditions for the convergence of solutions of impulse extension equations to the solutions of their associated impulsive differential equation are proven. Counterexamples are provided.

1. Introduction

Impulsive differential equations provide an elegant way to describe systems that undergo very fast changes in state [2,12,18]. These changes in state occur so quickly that they are idealized as discontinuities. Impulsive differential equations have a host of applications, including pulse vaccinations [1,8], seasonal skipping in recurrent epidemics [19], antiretroviral drug treatment [10,14] and birth pulses in animals [17].

Impulse extension equations have been put forward as a framework to study properties of impulsive differential equations that remain invariant if one replaces the impulse effect by a continuous perturbation [5]. Results on existence and uniqueness of solutions, as well as specialized results for linear periodic systems, have been developed [6,7].

In the present article, two similar but ultimately different problems are solved. First, given a linear impulsive differential equation, we associate to it a family of impulse extension equations that is parameterized by its step sequences. We then provide sufficient conditions under which the solutions of the impulse extension equation converge to those of the impulsive differential equation, as the step sequence becomes small. These

* Corresponding author.
E-mail address: rsmith43@uottawa.ca (R. Smith?).

http://dx.doi.org/10.1016/j.jmaa.2017.08.026

sufficient conditions are then tied to results relating to stability of the family of impulse extension equations, relative to the impulsive differential equation that generate it.

Following this, the *time-scale tolerance* is introduced first for linear, periodic impulsive differential equations, and then in general linear systems. The time-scale tolerance behaves as a robust stability threshold; if the norm of a given step sequence is smaller than the time-scale tolerance, than all impulse extensions equations from a particular class will have the same stability classification as the associated impulsive differential equation. From the point of view of applications, this indicates that if an impulsive differential equation models some physical process, then the approximation by an impulsive differential equation is, in a certain sense, "valid", provided the perturbations that are idealized as impulses occur on a time-scale that is smaller than the time-scale tolerance. Methods to compute the time-scale tolerance are proposed.

2. Background material on impulse extension equations

Throughout this paper, we will be interested in continuous systems that approximate the linear, finite-dimensional impulsive differential equation,

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), && t \neq \tau_k, \\ \Delta x &= B_k x + h_k, && t = \tau_k, \end{aligned} \tag{1}$$

as well as its associated homogeneous equation,

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, && t \neq \tau_k, \\ \Delta x &= B_k x, && t = \tau_k. \end{aligned} \tag{2}$$

It is assumed that the sequence of impulse times, $\{\tau_k\}$, is monotone increasing and unbounded. Also, we assume all functions appearing in the differential equations above are sufficiently regular to guarantee that for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, there is a unique solution $x(t)$ defined on $[t_0, \infty)$ satisfying $x(t_0) = x_0$. For example, it suffices to have all functions be bounded and measurable on compact sets.

We now comment on some notation related to sequences that will be relevant. If $s = \{s_n\}$ is a real-valued sequence, we define $\Delta s_n = s_{n+1} - s_n$ to be the forward difference. Also, indexed families of sequences, such as $\{s^j : j \in U\}$ for some index set U, will always have their index appear in the exponent. As such, the symbol s_n^j indicates the nth element of the sequence s^j, for $j \in U$.

The following definition of an *impulse extension equation* for (2) is a modified version of that appearing in [7]; the present definition is for linear systems, and allows us to more concretely study the convergence of their solutions, which is necessary to fulfill the objective of this article.

Definition 2.1. Consider the linear impulsive differential equation (1).

- A *step sequence over* τ_k is sequence of positive real numbers $a = \{a_k\}$ such that $a_k < \Delta\tau_k$ for all $k \in \mathbb{Z}$. We denote $\mathcal{S}_j = \mathcal{S}_j(a) \equiv [\tau_j, \tau_j + a_j)$ and $\mathcal{S} = \mathcal{S}(a) \equiv \bigcup_{j\in\mathbb{Z}} \mathcal{S}_j$. The set of all step sequences will be denoted $\mathcal{S}^*$, and is defined by

$$\mathcal{S}^* \equiv \{a : \mathbb{Z} \to \mathbb{R}_+ \,,\ 0 < a_k < \Delta\tau_k\}.$$

- A sequence of functions $\varphi = \{(\varphi_k^B, \varphi_k^h)\}$,

$$\varphi_k^B : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^{n\times n}, \qquad \varphi_k^h : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n,$$

is a *family of impulse extensions for* (1) if, for all $k \in \mathbb{Z}$ and all $a \in \mathcal{S}^*$, the functions $\varphi_k^\xi(\cdot, a_k)$ are locally integrable and satisfy the equality

$$\int_{\mathcal{S}_k(a)} \varphi_k^\xi(t, a_k)dt = \xi_k, \tag{3}$$

for $\xi \in \{B, h\}$.

- Given $a \in \mathcal{S}^*$ and a family of impulse extensions, φ, for the impulsive differential equation (1), the *impulse extension equation associated to* (1) *induced by* (φ, a) is the following differential equation with piecewise-constant arguments:

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t), && t \notin \mathcal{S}(a), \\ \frac{dx}{dt} &= A(t)x + g(t) + \varphi_k^B(t, a_k)x(\tau_k) + \varphi_k^h(t, a_k), && t \in \mathcal{S}_k(a). \end{aligned} \tag{4}$$

- To a homogeneous impulsive differential equation, (2), we can also consider the associated *homogeneous impulse extension induced by* (φ, a):

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, && t \notin \mathcal{S}(a), \\ \frac{dx}{dt} &= A(t)x + \varphi_k^B(t, a_k)x(\tau_k), && t \in \mathcal{S}_k(a). \end{aligned} \tag{5}$$

Definition 2.2. Let a family of impulse extensions, $\varphi = \{(\varphi_k^B, \varphi_k^h)\}$, and a step sequence $a \in \mathcal{S}^*$ be given. A function $y : I \to \mathbb{R}^n$ defined on an interval $I \subset \mathbb{R}$ is a *classical solution* of the impulse extension equation (4) induced by (φ, a) if y is continuous, the sets $I \cap S_k(a)$ are either empty or contain τ_k and y satisfies the differential equation (4) almost everywhere on I. Given an *initial condition*

$$x(t_0) = x_0, \tag{6}$$

with $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, the function $y(t)$ is a solution of the *initial-value problem* (4)–(6) if, in addition, $y(t_0) = x_0$. The notation $y(t; t_0, x_0, a)$ means that $y(\cdot) = y(\cdot; t_0, x_0, a)$ is a solution of the initial-value problem (4)–(6) with impulse extension equation induced by (φ, a).

Definition 2.3. The *predictable set* of an impulse extension equation (4) for (1) induced by (φ, a) is the set

$$\mathcal{P}(\varphi, a) = \mathbb{R} \setminus \left\{ t \in \overline{S(a)} : \det \left(I + \int_{\max_{\tau_k}\{\tau_k \leq t\}}^{t} X^{-1}(s, \tau_k)\varphi_k^B(s, a_k)ds \right) = 0 \right\}, \tag{7}$$

where $X(t, s)$ is the Cauchy matrix for the linear homogeneous ordinary differential equation $x' = A(t)x$.

The following proposition is a restatement of Lemma 4.2 of [7].

Proposition 2.1. *Consider an impulse extension equation for* (1) *induced by* (φ, a). *For* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, *the initial-value problem* (4)–(6) *with initial condition* $x(t_0) = x_0$ *has a unique solution defined on the interval* $I \subset \mathbb{R}$ *if and only if* $t_0 \in \mathcal{P}$ *and for all* $I \ni \tau_k + a_k < t_0$, *the inclusion* $\tau_k + a_k \in \mathcal{P}$ *holds.*

We will also make use of the following representation of *matrix solutions* of the homogeneous equation, (5).

Proposition 2.2. *Suppose $t_0 \in \mathcal{P}$. Then there exists a matrix-valued function, $U(\cdot;t_0) : [l,\infty) \to \mathbb{R}^{n\times n}$, with $l = \max_{\tau_k \le t_0} \tau_k$ satisfying $U(t_0;t_0) = I$, such that the unique solution of the initial-value problem $x(t_0) = x_0$ of the homogeneous initial-value problem, (5)–(6), for any $x_0 \in \mathbb{R}^n$, can be written as $x(t) = U(t;t_0)x_0$. In particular, we have the formula*

$$U(t;t_0) = \begin{cases} X(t;t_0), & t,t_0 \in (\tau_k + a_k, \tau_{k+1}] \\ X(t;\tau_j)L_a^j(t;\tau_j)\left[\prod_{r=j-1}^{k} X(\tau_{r+1};\tau_r)L_a^r(\tau_{r+1};\tau_r)\right]X(\tau_k;t_0) & \begin{array}{l} t_0 \in (\tau_{k-1} + a_{k-1}, \tau_k] \\ t \in [\tau_j, \tau_{j+1}],\ k < j \end{array} \\ U(t;\tau_k)U^{-1}(t_0;\tau_k), & t_0 \in (\tau_k, \tau_k + a_k) \end{cases} \tag{8}$$

where $X(t;s)$ is the Cauchy matrix of the homogeneous ordinary differential equation $x' = A(t)x$, and the function $L^k : (a,t) \mapsto L_a^k(t)$ is defined by

$$L_a^k(t) = I + \int_{\tau_k}^{\min\{t,\tau_k+a_k\}} X^{-1}(s;\tau_k)\varphi_k^B(s,a_k)ds. \tag{9}$$

3. Convergence properties of impulse extension equations

3.1. Convergence of solutions with respect to the step sequence as $a \to 0$

The main result of this section relates to the mode of convergence of solutions of the initial-value problem (4)–(6) with respect to the step sequence $a \in \mathcal{S}^*$.

Definition 3.1. Let a family φ of impulse extensions be given for an impulsive differential equation (1). Let $\sigma = \{\sigma_k\}$ be a sequence of positive real numbers, and let $w = \{(w_k^B, w_k^h)\}$ be a sequence of pairs of functions $w_k^\xi : [\tau_k, \tau_{k+1}] \times [0, \Delta\tau_k) \to \mathbb{R}$ with the property that w_k^ξ is continuous and vanishing at $(\tau_k, 0)$ and $w_k^\xi(\cdot, a)$ is integrable on $\mathcal{S}_k(a)$. We will say φ is (σ,w)-*exponentially regulated in the mean,* or simply (σ,w)-*regulated*, if

$$\varphi_k^\xi(t,s) - \frac{1}{s}\xi_k = \mathcal{O}\left(w_k^\xi(t,s)\frac{1}{e^{\sigma_k s} - 1}\right) \tag{10}$$

for $t \in [\tau_k, \tau_k + s)$ as $s \to 0$.

Lemma 3.1. *Let a homogeneous impulsive differential equation (5) be given. Let $X(t;s)$ be the Cauchy matrix of the homogeneous ordinary differential equation $x' = A(t)x$. Suppose for each $k \in \mathbb{Z}$, the inequality $||X(t;\tau_k)|| \le e^{\sigma_k(t-\tau_k)}$ holds for some $\sigma_k > 0$, for $t \in [\tau_k, \tau_{k+1}]$. If φ is a (σ,w)-regulated family of impulse extensions for (5), then $L_a^k \to I + B_k$ pointwise as $a \to 0$. If $N \subset [\tau_k, \tau_{k+1}]$ and no decreasing sequence in N converges to τ_k, then the convergence is uniform on N.*

Proof. First, notice that we can write L_a^k as

$$L_a^k(t;\tau_k) = I + \left[\frac{1}{a_k}\int_{\tau_k}^{m_a^k(t)} X^{-1}(s;\tau_k)ds\right]B_k + \int_{\tau_k}^{m_a^k(t)} X^{-1}(s;\tau_k)\epsilon_k^B(t,a_k)ds,$$

where $\epsilon_k^B(t,a_k) = \varphi_k^B(t,a_k) - \frac{1}{a_k}B_k$ and $m_a^k(t) = \min\{t, \tau_k + a_k\}$. Now let $a_k < t - \tau_k$, so we have $m_k^a(t) = \tau_k + a_k$. We then have

$$\frac{1}{a_k}\int_{\tau_k}^{m_k^a(t)} X^{-1}(s;\tau_k)ds - I = \frac{1}{a_k}\int_{\tau_k}^{\tau_k+a_k}\left[X^{-1}(s;\tau_k) - X^{-1}(\tau_k;\tau_k)\right]ds,$$

which clearly converges to zero as $a \to 0$, due to the continuity of $X^{-1}(s;\tau_k)$. Therefore we conclude that

$$\frac{1}{a_k}\int_{\tau_k}^{\tau_k+a_n^k} X^{-1}(s,\tau_k)ds \to I.$$

As for the other integral, we have $X^{-1}(s,\tau_k) = X(\tau_k, s)$. We thus have the estimation

$$\left|\left|\int_{\tau_k}^{\tau_k+a_k} X^{-1}(s,\tau_k)\epsilon_k^B(s,a_k)ds\right|\right| \leq \int_{\tau_k}^{\tau_k+a_k} e^{\sigma_k(\tau_k-s)}\frac{w_k^B(s,a_k)}{e^{\sigma_k a_k}-1}C_k ds \leq \frac{C_k}{\sigma_k}||w_k^B(\cdot,a_k)||,$$

for some constant $C_k > 0$, where we used the asymptotic condition (10) and Gronwall's inequality, and the norm on $w_k^B(\cdot,a_k)$ is the uniform norm over the interval $[\tau_k, \tau_k + a_k]$. Since w_k^B is continuous and vanishing at $(\tau_k, 0)$, we have that $||w_k^B(\cdot,a_k)|| \to 0$ as $a_k \to 0$. Therefore we conclude that the integral term converges to zero. It now follows that $L_a^k \to I + B_k$ pointwise as $a \to 0$.

The convergence is generally nonuniform because m_k^a does not converge uniformly. However, the convergence can be made uniform on $N \subset [\tau_k, \tau_{k+1}]$ if τ_k is not an accumulation point of N. Take $a_k < \inf N - \tau_k$ so that we have $t > \tau_k + a_k$ for all $t \in N$, from which it follows that $m_k^a(t) = \tau_k + a_k$ on N. Then, the previous argument proceeds without modification, but the result holds uniformly for $t \in N$. □

Theorem 3.1. *For any linear impulsive differential equation (4), there exists a sequence of positive real numbers, $\sigma = \{\sigma_k\}$, such that, for any (σ, w)-regulated family φ of impulse extensions for (4), the following are true.*

- *For all $t_0 \in \mathbb{R}$, there exists $\delta = \delta(t_0) > 0$, such that, for $a \in \mathcal{S}^*$ with $||a||_\infty < \delta$ and all $x_0 \in \mathbb{R}^n$, the impulse extension equation (4) induced by (φ, a) possesses a unique classical solution, $x(t; t_0, x_0, a)$, satisfying the initial condition $x(t_0; t_0, x_0, a) = x_0$, and is defined for $t \geq t_0$.*
- *If $\det(I + B_k) \neq 0$ for all $k \in \mathbb{Z}$, the function $t \mapsto x(t; t_0, x_0, a)$ converges pointwise to $x(t; t_0, x_0, 0)$, the solution of the initial-value problem $x(t_0) = x_0$ for the impulsive differential equation, (1), as $||a||_\infty \to 0$.*
- *If $N \subset \mathbb{R}$ is bounded and no strictly decreasing sequence in N has an impulse time τ_k as its limit, the above convergence is uniform for $t \in N$ as $a \to 0$.*

In particular, it suffices to choose

$$\sigma_k = \int_{\tau_k}^{\tau_{k+1}} ||A(s)||ds. \tag{11}$$

Proof. Throughout this proof, φ is a fixed family of impulse extensions for (1). The first part of the Theorem is trivial. If $t_0 = \tau_k$ for some k, then we have $\tau_k \in \mathcal{P}$ by definition. Conversely, if $t_0 \neq \tau_k$ but $t_0 \in (\tau_k, \tau_{k+1})$, then as long as we have $a_k < t_0 - \tau_k = \delta(t_0)$, we will have $t_0 \in \mathcal{P}$. It now follows by Proposition 2.1 that, for all $x_0 \in \mathbb{R}^n$, there exists a unique solution of the initial-value problem $x(t_0) = x_0$ defined on $[t_0, \infty)$ provided $||a||_\infty < \delta(t_0)$. We will now denote the solution $x(t; t_0, x_0, a)$ to indicate the dependence on x_0 and a.

Next, we will prove that $x(t; t_0, x_0, a) \to x(t; t_0, x_0, 0)$ uniformly for $t \in N$ as $||a||_\infty \to 0$, as stated in the third conclusion of the theorem. We will only prove the uniform convergence result, since this implies pointwise convergence everywhere.

Before we begin, note that it suffices to prove the convergence on a compact, connected interval. Indeed, if the convergence is uniform on the closure of N, then it is uniform on N itself, and if N is disconnected, then it must be contained in a finite union of connected intervals $N_1, \dots, N_n$, each of which has the property of not having any impulse time as a left limit point. Therefore we will assume that the compact set N is a closed interval contained in $(\tau_k, \tau_{k+1}]$ for some k.

Let $||a||_\infty < \delta(t_0)$. Since $t_0 \in \mathcal{P}$, it follows by Proposition 4.2 of Church and Smith? [7] that we can write

$$x(t; t_0, x_0, a) = U(t; t_0, a)x_0 + x^p(t, a),$$

where $U(t; t_0, a)$ is the matrix solution for homogeneous equation of (4) with step sequence a and satisfying $U(t_0; t_0, a) = I$, and $x^p(t, a)$ is the solution of the inhomogeneous equation (4) with step sequence a and satisfying $x^p(t_0, a) = 0$. By this decomposition, it suffices to prove the uniform convergence of $U(t; t_0, a)$ and $x^p(t, a)$.

We first demonstrate the convergence of $U(t; t_0, a)$. For simplicity, we will assume $t_0 = \tau_0$; the other cases follow by similar reasoning, due to the representation provided by equation (8). For $t \in N \subset (\tau_k, \tau_{k+1}]$, we have

$$U(t, a) = X(t; \tau_k)L_a^k(t; \tau_k) \prod_{r=k-1}^{0} X(\tau_{r+1}; \tau_r)L_a^k(\tau_r + a_r; \tau_r),$$

where we have suppressed the dependence on t_0. Therefore,

$$||U(t, a) - U(t, 0)|| \le \sup_{t \in N} ||X(t; \tau_k)|| \cdot \Bigg|\Bigg| \left(L(t; \tau_k) \prod_{r=k-1}^{0} X(\tau_{r+1}; \tau_r)L(\tau_r + a; \tau_r) \right) \tag{12}$$

$$- \; (I + B_k) \left(\prod_{r=k-1}^{0} X(\tau_{r+1}; \tau_r)(I + B_r) \right) \Bigg|\Bigg|. \tag{13}$$

It suffices to prove that if $a \to 0$, then $L_a^j(t; \tau_j) \to I + B_k$ uniformly on N for each $j = 0, \dots, k$.

By the generalized Gronwall's inequality, we will have $||X(t; \tau_j)|| \le e^{\sigma_k(t-\tau_j)}$ for $t \in [\tau_j, \tau_{j+1}]$, where the constants σ_j are given by (11). Define $\sigma = \{\sigma_k\}$. If φ is (σ, w)-regulated, Lemma 3.1 guarantees the required convergence of $L_a^j(t; \tau_j)$ for $j = 0, \dots, k$, uniformly on N. We conclude that $U(t, a) \to U(t, 0)$ uniformly on N.

Next we show that $x^p(t, a)$ converges to the associated impulsive solution. This will be established by induction on the structure of the compact set N.

Similarly to before, we will assume that $t_0 \in [\tau_0, \tau_1]$. Other cases follow by similar reasoning. Suppose $N = [N^-, N^+] \subset (\tau_0, \tau_1]$. Then we have

$$x^p(t, a) = X(t; \tau_0) \int_{\tau_0}^{t} X^{-1}(s; \tau_0)(g(s) + \varphi_0^h(s, a_0) \cdot \mathbb{1}[\mathcal{S}_0(a)])ds + X(t; \tau_0)L_a^0(t; \tau_0)x_0^p(a),$$

$$x_0^p(a) = -L_{a_0}^{-1}(t_0; \tau_0) \int_{\tau_0}^{t_0} X^{-1}(s; \tau_0) \left(g(s) + \varphi_0^h(s, a) \cdot \mathbb{1}[\mathcal{S}_0(a)] \right) ds,$$

$$x^p(t,0) = X(t;\tau_0)\left[(I+B_0)x_0^p(0) + h_0 + \int_{\tau_0}^{t} X^{-1}(s;\tau_0)g(s)ds + h_0\right],$$

$$x_0^p(0) = -(I+B_0)^{-1}\left[h_0 + \int_{\tau_0}^{t_0} X^{-1}(s;\tau_0)g(s)ds\right].$$

Note that if $||a||_\infty$ is sufficiently small, then $x_0^p(a)$ is well-defined, since $(L_a^0)^{-1}(t_0;\tau_0) \to (I+B_0)^{-1}$ by Lemma 3.1.

We then have the equality

$$\begin{aligned} x^p(t,a) - x^p(t,0) = X(t;\tau_0)&\left[\int_{\tau_0}^{\tau_0+a_0} X^{-1}(s;\tau_0)\left(\varphi_0^h(s,a_0) - X(s;\tau_0)\frac{1}{a_0}h_0\right)ds\right] \\ &+ X(t;\tau_0)\left(L_a^0(t;\tau_0)x_0^p(a) - (I+B_0)x_0^p(0)\right], \end{aligned} \tag{14}$$

provided $a_0 < N^- - \tau_0$. A routine application of the triangle inequality, the (σ, w)-regularity condition and an argument similar to the proof of Lemma 3.1 can be used to show the integral converges to zero as $a \to 0$. We also have $L_a^0(t;\tau_0) \to I + B_0$ uniformly, so it suffices to prove the convergence $x_0^p(a) \to x_0^p(0)$.

We perform a similar bracketing, obtaining the formula

$$\begin{aligned} x_0^p(a) - x_0^p(0) = &-\left(L_a^0(t_0;\tau_0) - I - B_0\right)\int_{\tau_0}^{t_0} X^{-1}(s;\tau_0)g(s)ds \\ &- \int_{\tau_0}^{\tau_0+a_0} X^{-1}(s;\tau_0)\left[L_a^0(t_0;\tau_0)\varphi_0^h(s,a) - (I+B_0)\frac{1}{a_0}h_0\right]ds, \end{aligned}$$

which is valid if $a_0 < t_0 - \tau_0$. The first term clearly converges to zero, and the second one can be shown to converge to zero as well, in the same way as the convergence of the integral in equation (14). Therefore $x_0^p(a) \to x_0^p(0)$ as $a \to 0$, from which we obtain the convergence $x_0^p(t,a) \to x_0^p(t,0)$ as $a \to 0$, uniformly for $t \in N$.

For the induction hypothesis, assume that if $N \subset= [N^-, N^+] \subset (\tau_{k-1}, \tau_k]$ for some $k \geq 0$, then $x^p(t,a) \to x^p(t,0)$ as $a \to 0$, uniformly for $t \in \mathbb{N}$. Without loss of generality, we may assume that $N^+ = \tau_k$. Now if $M = [M^-, M^+] \subset (\tau_k, \tau_{k+1}]$, we have

$$\begin{aligned} x^p(t,a) &= X(t;\tau_k)L_a^k(t;\tau_k)x^p(\tau_k,a) \\ &\quad + X(t;\tau_k)\int_{\tau_k}^{t} X^{-1}(s;\tau_k)(g(s) + \varphi_k^h(s,a_k)\cdot \mathbb{1}[\mathcal{S}_k(a)])ds, \\ x^p(t,0) &= X(t;\tau_k)\left[(I+B_k)x^p(\tau_k,0) + h_k + \int_{\tau_k}^{t} X^{-1}(s;\tau_k)g(s)ds + h_k\right). \end{aligned}$$

Subtracting, we obtain

$$\begin{aligned} x^p(t,a) - x^p(t,0) = X(t;\tau_k)&\left[L_a^k(t;\tau_k)x^p(\tau_k,a_k) - [E+B_k]x^p(\tau_k,0)\right] \\ &+ X(t;\tau_k)\left[\int_{\tau_k}^{\tau_k+a_k} X^{-1}(s;\tau_k)\varphi_k^h(s,a_k) - \frac{1}{a_k}h_k\right], \end{aligned}$$

provided $a_k < M^- - \tau_0$. By the induction hypothesis, we have $x^p(\tau_k, a) \to x^p(\tau_k, 0)$. By Lemma 3.1, we know that $L_a^k(t; \tau_k) \to I + B_k$. By the same reasoning as earlier, the integral term converges to zero. Therefore we conclude that $x^p(t, a) \to x^p(t, 0)$ on M. Hence, for any k, if N is a closed interval in $(\tau_k, \tau_{k+1}]$, $x^p(t, a)$ converges uniformly to $x^p(t, 0)$ as $a \to 0$. This proves the theorem. □

Corollary 3.1.1. *The conclusions of Theorem 3.1 hold if, for all $k \in \mathbb{Z}$ and $\xi \in \{B, h\}$, the relation*

$$\varphi_k^\xi(t, s) - \frac{1}{s}\xi_k = O(1) \tag{15}$$

holds for $t \in [\tau_k, \tau_k + s)$ as $s \to 0$.

Proof. The choice of functions $w_k^\xi(t, s) = e^{\sigma_k s} - 1$ satisfies the conditions of the theorem and provides the asymptotic relation (15). □

Corollary 3.1.2. *If φ is (σ, w)-regulated family of impulse extensions for (4) and*

$$\sigma = \max_{k\in\mathbb{Z}} \int_{\tau_k}^{\tau_{k+1}} ||A(s)||ds \tag{16}$$

is finite, the conclusions of Theorem 3.1 hold.

3.2. Convergence of Floquet multipliers for periodic systems with respect to the step sequence as $a \to 0$

In this section, we assume that $\det(I + B_k) \neq 0$ for all $k = 0, \dots, c-1$. One consequence of Corollary 3.1.2 is that the Floquet multipliers of a periodic, homogeneous impulse extension equation, (5), induced by an (φ, a), will converge to those of the associated periodic, homogeneous impulsive differential equation, as $a \to 0$. To make this precise, we first state two definitions.

Definition 3.2. The impulsive differential equation (1) is *periodic with period T and cycle number c*, or *(T, c)-periodic,* if A and f are T-periodic and c is the smallest natural number such that the shift identities $\tau_{k+c} = \tau_k + T$, $B_{k+c} = B_k$ and $h_{k+c} = h_k$ for all $k \in \mathbb{Z}$.

Definition 3.3. An step sequence $a \in \mathcal{S}^*$ is *c-periodic* if $a_{k+c} = a_k$ for all $k \in \mathbb{Z}$. Denote by $\mathcal{S}_c^*$ the set of c-periodic step sequences. If equation (4) is (T, c)-periodic, a family of impulse extensions, φ, is *(T, c)-periodic* if the shift property

$$\varphi_{k+c}^\alpha(t + T, a) = \varphi_k^\alpha(t, a)$$

holds for $a \in \mathcal{S}_c^*$, $t \in \mathcal{S}_k(a)$ and integers k, where $\alpha \in \{B, h\}$.

The above definitions imply that a finite set of functions $\varphi = \{(\varphi_k^B, \varphi_k^h)\}$ suffices to define a (T, c)-periodic family of impulse extensions. As such, we will say that a (T, c)-periodic family φ is (σ, w)-regulated if the asymptotic condition (10) is satisfied for indices $k = 0, \dots, c-1$. As such, the sequences σ and w can also be taken as c-element indexed sets.

Analogous definitions to the above hold for homogeneous equations, for which the following corollary is relevant.

Corollary 3.1.3. *Let the homogeneous impulsive equation (2) be (T, c)-periodic. Let φ be a (T, c)-periodic family of impulse extensions for (5), and suppose φ is exponentially (σ, w)-regulated in the mean with*

$$\sigma = \sigma_A \equiv \max_{k=0,\ldots,c-1} \int_{\tau_k}^{\tau_{k+1}} ||A(s)||ds. \tag{17}$$

If M_a denotes the monodromy matrix of the impulse extension equation induced by (φ, a), and M_0 is the monodromy matrix of the impulsive differential equation (2), we have $M_a \to M_0$ as $a \to 0$, where the convergence is for $a \in \mathcal{S}_c^*$. The result remains valid if $\sigma = \sigma_F \equiv ||\Lambda||$, where $X(t) = \Phi(t)e^{\Lambda(t-\tau_0)}$ is the Floquet factorization of the homogeneous equation $z' = A(t)z$, and Φ is T-periodic and satisfies $\Phi(\tau_0) = I$.

Proof. By theorem from Church and Smith? [7], we have $M_a = U(T + \tau_0; \tau_0, a)$, where $U(t; \tau_0, a)$ is the matrix solution of (5) with impulse extension family φ and step sequence $a \in \mathcal{S}_c^*$, satisfying $U(\tau_0; \tau_0, a) = I$. By Corollary 3.1.2, we have $U(T+\tau_0; \tau_0, a) \to U(T+\tau_0; \tau_0, 0) = M_0$ as $a \to 0$. If $\sigma = ||\Lambda||$, we note that one has $||X(t)|| \le Ke^{\sigma(t-\tau_0)}$, where $K = \sup_{t\in[\tau_0,\tau_0+T]}$, from which the result follows due to Lemma 3.1. □

Since the stability or instability of a homogeneous impulse extension equation can be inferred from the spectrum of the monodromy matrix [7], the above corollary implies that, unless the spectrum of M_0 intersects the unit circle, the stability of the impulsive differential equation will match that of the impulse extension equation induced by (φ, a) provided $||a||$ is small enough. We have the following corollary whose proof we omit, since it follows from the above results and Corollary 5.2 of [7].

Corollary 3.1.4. *Let M_0 denote the monodromy matrix of the (T, c)-periodic impulsive equation (2). If $\sigma(M_0)$ does not intersect the unit circle, (2) is asymptotically stable if and only if, for all (σ, w)-regulated families of impulse extensions, φ, with σ as defined in Corollary 3.1.3, there exists $\delta > 0$ such that, for all $||a|| < \delta$, the impulse extension equation for (2) induced by (φ, a) is stable for any $t_0 \in \mathcal{P}(\varphi, a)$.*

3.3. Asymptotic stability of aperiodic systems as the step sequence becomes small, $a \to 0$

If the impulsive linear system (2) is stable but not asymptotically stable, then nothing can in general be said for the stability for associated impulse extension equations, (5). In fact, asymptotic stability is, in general, required, as the example from Section 3.4.2 illustrates.

The main result of this section rests on the following two lemmas.

Lemma 3.2. *Let x_n be a bounded real-valued sequence, and suppose the inequality $\prod_{i=s_n}^{s_{n+1}} x_i \le C$ holds for all $n \in \mathbb{N}$ and some $C \in (0, 1)$ and a monotone sequence of natural numbers, s_n, with bounded finite difference, for which $s_n \to \infty$. Then there exists $\epsilon^* > 0$ such that, for all $\epsilon \in [0, \epsilon^*)$, the inequality*

$$\prod_{i=s_n}^{s_{n+1}} (x_i + \epsilon) \le C(\epsilon)$$

holds for some $C(\epsilon) \in (0, 1)$, for all $n \in \mathbb{N}$.

Proof. Let $\Delta s_n \le D$ be the upper bound on the finite difference and $|x_n| \le X$ be an upper bound. For each $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$, we have

$$\begin{aligned}\prod_{i=s_n}^{s_{n+1}} (x_i + \epsilon) &\le \prod_{i=s_n}^{s_{n+1}} x_i + \sum_{i=1}^{\Delta s_n} \binom{\Delta s_n}{i} \epsilon^i X^{\Delta s_n+1-i} + \prod_{i=s_n}^{s_{n+1}} \epsilon \\ &\le C + \sum_{i=1}^{\Delta s_n} \epsilon^i \max\{X, 1\}^{D+1} + \epsilon^{D+1}\end{aligned}$$

$$\leq C + \sum_{i=1}^{D} i\epsilon \max\{X,1\}^{D+1} + (D+1)\epsilon$$
$$= C + \epsilon\left(\frac{1}{2}D(D+1)\max\{X,1\}^{D+1} + D + 1\right) \equiv C(\epsilon).$$

Define ϵ^* by

$$\epsilon_1^* = \frac{1-C}{\frac{1}{2}D(D+1)\max\{X,1\}^{D+1} + D + 1}.$$

It follows that if $0 \leq \epsilon < \min\{\epsilon_1^*, 1\} \equiv \epsilon^*$, then $C(\epsilon) < 1$. □

Lemma 3.3. *Let x_n be a bounded, nonnegative, real-valued sequence, and suppose the inequality $\prod_{i=s_n}^{s_{n+1}} x_i \leq C$ holds for all $n \in \mathbb{N}$ and some $C \in (0,1)$ and a monotone sequence of natural numbers, s_n, with bounded finite difference, for which $s_n \to \infty$. The infinite product $\prod_{i=0}^{\infty} x_i$ diverges to zero.*

Proof. Without loss of generality, suppose $s_0 = 0$. As before, let $\Delta s_n \leq D$ be the upper bound on the finite difference and $|x_n| \leq X$ be an upper bound. By construction, there exists a subsequence of s_n, denoted s_{n_m}, and a bounded sequence $d_m \in \{0,1,\dots,D-1\}$ such that $m = s_{n_m} + d_m$ for all $m \in \mathbb{N}$.

Consider $|p_m| = \prod_{i=0}^{m} |x_i|$, the modulus of the sequence of partial products. By the above representation of $m \in \mathbb{N}$, we have

$$|p_m| = \left(\prod_{i=0}^{s_{n_m}} |x_i|\right) \prod_{i=s_{n_m}+1}^{s_{n_m}+d_m} |x_i| \leq \left(\prod_{j=0}^{m}\prod_{i=s_j}^{s_{j+1}} |x_i|\right) \prod_{i=s_{n_m}+1}^{s_{n_m}+d_m} |x_i| \leq C^{m+1}\max\{1,X\}^D.$$

Since $C < 1$ and X and D are finite, we have $|p^m| \to 0$ as $m \to \infty$. □

Theorem 3.2. *Let a homogeneous impulsive system (2) be given. Define the sequences C_i and D_i by*

$$D_i = \int_{\tau_i}^{\tau_{i+1}} ||A(s)||^2 \exp\left(-2\int_{\tau_i}^{s} ||A(r)||dr\right) ds, \tag{18}$$
$$E_i = ||I + B_i||(1 + \sqrt{(\Delta\tau_i)D_i}). \tag{19}$$

Consider the following conditions.

A1: *The sequences D_i and E_i are bounded.*

A2: *There exists a strictly increasing sequence of natural numbers s_n with a bounded forward difference and a real number $C \in (0,1)$ such that, for all $n \in \mathbb{N}$,*

$$\prod_{i=s_n}^{s_{n+1}} E_i \leq C. \tag{20}$$

A3: *$\varphi = \{\varphi_k\}$ is a family of impulse extensions for (2) that satisfies the asymptotic relations*

$$\left|\left|\varphi_k(t,a) - \frac{1}{a}B_k\right|\right| \leq g(t,a), \qquad \int_{\mathcal{S}_k(a)} h(s;\tau_k) + \frac{1}{2}h(s;\tau_k)^2 ds \leq G(a), \qquad h(s;\tau_k) = \int_{\tau_k}^{s} g(r,a)dr \tag{21}$$

on $\mathcal{S}_k(a)$ as $||a||_\infty \to 0$, for some function G satisfying $G(a) \to 0$ as $||a||_\infty \to 0$.

A4: *φ is (σ, w)-regulated in the mean with $||X^{-1}(t;\tau_k)|| \le e^{\sigma_k(t-\tau_k)}$ for $t \in [\tau_k, \tau_{k+1}]$ for all $k \in \mathbb{Z}$, where $X(t;s)$ is the Cauchy matrix of $x' = A(t)x$.*
A5: *$\det(I + B_k) \neq 0$ for all $k \in \mathbb{Z}$.*

System (2) is asymptotically stable, and, for all $t_0 \in \mathbb{R}$, there exists $\delta > 0$ such that if $||a||_\infty < \delta$, the impulse extension equation for (2) induced by (φ, a) is asymptotically stable at t_0 and uniformly attracting on $\mathbb{R}$. If $N \subseteq \mathcal{P}(\varphi, a)$ is bounded and separated from $\mathbb{R} \setminus \mathcal{P}(\varphi, a)$, the previous result holds with uniform asymptotic stability on N.

Proof. Let $t_0 \in \mathbb{R}$. If $||a||$ is sufficiently small, then, by definition, $t_0 \in \mathcal{P}(\varphi, a)$, so we may assume $t_0 \in \mathcal{P}(\varphi, a)$. We prove only the case of $t_0 = \tau_0$; the other cases follow by similar reasoning (with the only significantly different cases being if t_0 is in the interior of $\mathcal{S}_k(a)$ for some k; in this instance, formula (8) is useful). By the generalized Gronwall's inequality, any matrix solution, $U(t)$, of (5), for which $U(t_0) = I$, satisfies the inequality

$$|U(t)| \le |U(\tau_k)| \left(G_k(t) + \int_{\tau_k}^{t} G_k(s)|A(s)| \exp\left(-\int_{t}^{s} |A(r)|dr \right) ds \right),$$

for $t \in [\tau_k, \tau_{k+1}]$, where $|\cdot|$ denotes the standard Euclidean norm (or induced matrix norm),

$$G_k(t) = \left| I + \int_{\tau_k}^{t} \varphi_k(s, a)ds \right|,$$

and we identifying φ_k with $\varphi_k \cdot \mathbb{1}[\mathcal{S}_k(a)]$. By a simple inductive argument, we can see that, for $t \in [\tau_k, \tau_{k+1}]$,

$$|U(t)| \le |U(\tau_0)| F_k(t) \prod_{i=0}^{k-1} \left(G_i(\tau_{i+1}) + \int_{\tau_i}^{\tau_{i+1}} G_s(a)|A(s)| \exp\left(-\int_{\tau_i}^{s} |A(r)|dr \right) ds \right), \tag{22}$$

$$F_k(t) = G_k(t) + \int_{\tau_k}^{t} G_k(s)|A(s)| \exp\left(-\int_{\tau_k}^{s} |A(r)|dr \right) ds. \tag{23}$$

An overestimate of $F_k(t)$ can be obtained via the Cauchy–Schwarz inequality, together with maximizing the integral by taking the upper limit as τ_{k+1}. We have

$$F_k(t) \le 1 + |B_k| + \left(\int_{\tau_k}^{\tau_{k+1}} G_k(s)^2 ds \right)^{\frac{1}{2}} \underbrace{\left(\int_{\tau_k}^{\tau_{k+1}} |A(s)|^2 \exp\left(-2\int_{\tau_k}^{s} |A(r)|dr \right) ds \right)^{\frac{1}{2}}}_{D_k}.$$

Since $G_k(t)$ is constant and equal to $|I + B_k|$ on $[\tau_{k+a_k}, \tau_{k+1}]$, we can write

$$F_k(t) \le 1 + |B_k| + \left(\int_{\mathcal{S}_k(a)} G_k(s)^2 ds + (\Delta\tau_k - a_k)|I + B_k|^2 \right)^{\frac{1}{2}} D_k^{\frac{1}{2}}. \tag{24}$$

We now have the task of estimating the integral of G_k^2 on $\mathcal{S}_k(a)$. Using the asymptotic condition (21), we can write

$$|\varphi_k(t)| \leq \frac{1}{a_k}|B_k| + g(t,a)$$

provided $||a||_\infty$ is sufficiently small, for all $t \in \mathcal{S}_k(a)$. It follows that

$$G_k(t) \leq 1 + \int_{\tau_k}^{s} \left(\frac{1}{a_k}|B_k| + g(s,a)\right) ds. \tag{25}$$

Substituting the upper bound from (25) into (24), we obtain

$$\begin{aligned} F_k(t) &\leq 1 + |B_k| + \left(\int_{\mathcal{S}_k(a)} 1 + 2\int_{\tau_k}^{s} \frac{1}{a_k}|B_k| + g(r,a)dr + \left[\int_{\tau_k}^{s} \frac{1}{a_k}|B_k| + g(r,a)dr\right]^2 ds \right)^{\frac{1}{2}} D_k^{\frac{1}{2}} \\ &\leq 1 + |B_k| + \left(a_k\left(1 + |B_k| + \frac{1}{2}a_k|B_k|^2\right) + 2\int_{\mathcal{S}_k(a)} \int_{\tau_k}^{s} g(r,a)drds + \int_{\mathcal{S}_k(a)} \left[\int_{\tau_k}^{s} g(r,a)dr\right]^2 ds \right)^{\frac{1}{2}} D_k^{\frac{1}{2}} \\ &\leq 1 + |B_k| + \left(a_k\left(1 + |B_k| + \frac{1}{2}a_k|B_k|^2\right) + 2G(a)\right)^{\frac{1}{2}} D_k^{\frac{1}{2}}, \end{aligned}$$

where in the last line we used the integral estimate in (21). By hypothesis, the sequence D_k is bounded. Boundedness of D_k implies the boundedness of B_k, from which we conclude

$$F_k(t) \leq 1 + B + \underbrace{\left(a_k\left(1 + B + \frac{1}{2}a_kB^2\right) + 2G(a)\right)^{\frac{1}{2}} D^{\frac{1}{2}}}_{R_k}$$

for positive constants B and D with $D_i \leq D$. It follows that $R_k \to 0$ as $||a||_\infty \to 0$, uniformly for all $k \in \mathbb{N}$.

By similar arguments, one can show that the bound

$$|U(t)| \leq |U(\tau_0)|(1 + B + R_k) \prod_{i=0}^{k-1} |I + B_i| + \sqrt{(\Delta\tau_i - a_i)|I + B_i|^2 + R_i^2} \cdot \sqrt{D_i}$$

holds for all $t \in [\tau_k, \tau_{k+1}]$. Since $R_k \to 0$ uniformly for $k \in \mathbb{N}$ as $||a||_\infty \to 0$ and the D_i are independent of a, we can, for any $\epsilon > 0$ small, ensure that

$$\begin{aligned} |U(t)| &\leq |U(\tau_0)|(1 + B + \epsilon) \prod_{i=0}^{k-1} |I + B_i| + \sqrt{(\Delta\tau_i - \epsilon)|I + B_i|^2 + \epsilon^2} \cdot \sqrt{D_i} \\ &\leq |U(\tau_0)|(1 + B + \epsilon) \prod_{i=0}^{k-1} |I + B_i|(1 + \sqrt{(\Delta\tau_i - \epsilon)D_i}) + \epsilon\sqrt{D} \\ &\leq |U(\tau_0)|(1 + B + \epsilon) \prod_{i=0}^{k-1} \left[E_i + \epsilon\sqrt{D}\right], \end{aligned}$$

for $t \in [\tau_k, \tau_{k+1}]$ and all $k \in \mathbb{N}$, by choosing $||a||_\infty$ small enough. By Lemma 3.2 and Lemma 3.3, there exists some $\epsilon^* > 0$ such that the infinite product $\prod_{i=0}^{\infty}[E_i + \epsilon\sqrt{D}]$ diverges to zero, provided $\epsilon < \epsilon^*$. It follows that $|U(t)| \to 0$ when $\epsilon < \epsilon^*$, which is equivalent to $||a||_\infty < \delta$ for some sufficiently small δ. This also proves uniform attractivity on $\mathbb{R}$; if $x(t)$ and $y(t)$ are two solutions defined for $t \geq t^*$, then there exists $\tau_k \geq t^*$, and, by Lemma 4.3 of [7], we have $x(t) - y(t) = U(t)U^{-1}(\tau_k)(x(\tau_k) - y(\tau_k)) \to 0$ as $t \to \infty$.

Next, since any solution of the initial-value problem $x(t_1) = x_1$ for (5) with $t_1 \in N$ can be written as $x(t) = U(t)U^{-1}(t_1)x_1$, we obtain $|x(t)| < \eta$ for all $t \geq t_0$ provided

$$|x_1| < \frac{\eta}{\sup_{t \geq t_0} |U(t)| \cdot |U^{-1}(t_1)|},$$

where the supremum exists due to boundedness of U for $t \geq t_0$, and $U^{-1}(t_1)$ exists since $t_1 \in N \subset \mathcal{P}(\varphi, a)$; see [7] for more details. Therefore (5) is stable and attracting on N, and so is asymptotically stable.

If N is bounded and separated from $\mathbb{R} \setminus \mathcal{P}$, it follows that $\bar{t} \equiv \inf N \in \mathcal{P}(\varphi, a)$ and that $K \equiv \sup_{t \in N} |U^{-1}(t)|$ and $J \equiv \sup_{t \in N} |U(t)|$ are finite (provided $||a||$ is chosen sufficiently small so as to guarantee that $\tau_k + a_k \in \mathcal{P}(\varphi, a)$ for all $\tau_k \in N$; this can always be done because N is bounded and assumptions A4–A5 hold; see Lemma 3.1 and Theorem 4.2 of [7]). Then, replacing the bound above with $|x_1| < \eta/(JK)$, we obtain uniform stability on N. □

3.4. Counterexamples

Some of the previously stated results are, in a certain sense, optimal, while others are not. The counterexamples of this section appear in, or are inspired by counterexamples appearing in [5].

3.4.1. (σ, w)-regularity is sufficient, but not necessary, for pointwise convergence of solutions

Consider the simple scalar equation

$$\begin{aligned} x' &= x, & t &\neq k \\ \Delta x &= -0.75x, & t &= k, \end{aligned} \tag{26}$$

with $k \in \mathbb{Z}$. This equation is periodic with period one, and its Floquet multiplier is $\mu_1 = \frac{1}{4}e$, which is less than one. Consequently, the trivial solution is asymptotically stable.

Consider now a periodic impulse extension for (26):

$$\begin{aligned} x' &= x, & t &\notin [k, k+a) \\ x' &= x + \varphi(t-k, a), & t &\in [k, k+a), \end{aligned} \tag{27}$$

where $\varphi(\cdot, a) : [0, 1] \to \mathbb{R}$ (note that we are taking advantage of the fact that, since (26) is periodic with order one, a family of impulse extensions is generated by a single function). For fixed $a \in (0, 1)$, the solution of (27) satisfying the initial condition $x(0; a) = 1$ is given by

$$x(t; a) = e^t \left(1 + \int_0^t e^{-s} \varphi(s, a) \cdot \mathbb{1}[0, a] ds \right)$$

for $t \in [0, 1]$. In particular, if we set $g(s, a) = \varphi(s, a) - \frac{1}{a}(-0.75)$, we have the equality

$$x(1; a) = e \left(1 + \int_0^a e^{-t} \frac{-0.75}{a} dt + \int_0^a e^{-t} g(t, a) ds \right).$$

In the limit, as $a \to 0^+$, we have

$$x(1;a) \to \frac{1}{4}e + e \underbrace{\lim_{a\to 0^+} \int_0^a e^{-t} g(t,a)dt}_{R(g)}.$$

Therefore pointwise convergence of the solution at time $t = 1$ is equivalent to having $R(g) = 0$.
If $g(t,a) = \frac{1}{e^a-1}\sin\left(\frac{2\pi t}{a}\right)$, then computing $R(g)$ gives

$$R(g) = \lim_{a\to 0^+} \int_0^a e^{-t}\frac{1}{e^a - 1}\sin\left(\frac{2\pi t}{a}\right)dt = \lim_{a\to 0^+} \frac{2\pi a e^{-a}}{a^2+4\pi^2} = 0.$$

For the linear impulsive equation (26), we have $\sigma_A = 1$, but this particular choice of $g(t,a)$ does not satisfy the $(1,w)$-regularity requirement, (10), for any w. Consequently, $\varphi(t,a) = \frac{1}{a}(-0.75) + g(t,a)$ is not $(1,w)$-regulated, but we do see pointwise convergence of the solutions at $t = 1$. One can clearly see that this holds for all $t \in [0,1]$; by periodicity, we obtain pointwise convergence everywhere.

On the other hand, if we choose

$$h(t,a) = \frac{a^2+4\pi^2}{2\pi a(1-e^{-a})}\sin\left(\frac{2\pi t}{a}\right),$$

we obtain $R(h) = 1$, and h also fails the $(1,w)$-regularity requirement. It is also far more singular at $a = 0$ than is g, but this is beside the point. The usefulness of the definition of (σ,w)-regularity stems from the fact that it does require a specific functional form of the solution of any given differential equation to be applied, as illustrated by Corollary 3.1.1. In this counterexample, the general solution of the homogeneous equation is expressible analytically, allowing for a more precise condition on pointwise convergence of solutions to be stated.

3.4.2. Corollary 3.1.4 does not hold in the presence of unit Floquet multipliers of the impulsive system

Consider the "trivially impulsive" impulsive differential equation

$$\begin{aligned} \frac{dr}{dt} &= r\sin(t), & t &\neq 2k\pi, \\ \Delta r &= 0, & t &= 2k\pi. \end{aligned} \tag{28}$$

The Floquet multiplier of this system is $\mu_0 = 1$, and the fundamental matrix solution at $t_0 = 0$ is $X(t) = \exp(1 - \cos(t))$. Therefore Corollary 3.1.4 cannot be applied. Let us consider for any $a \in (0, 2\pi)$, the family of impulse extensions

$$\varphi(t,a) = a^5 \sin\left(\frac{2\pi t}{a}\right)\sin\left(\frac{1}{a}\right)$$

with $\varphi(t,0) \equiv 0$ for all t. We have $||\varphi(t,a)|| \leq a^5$, so that φ is (σ,w)-regulated for any σ and $w = (e^{\sigma a} - 1)a^5$.

Note that

$$c(a) \equiv \int_0^a e^{\cos(t)}\sin\left(\frac{2\pi t}{a}\right)dt > 0$$

for $a < \pi$. We argue this as follows. For $0 \leq t < \pi$, the function $e^{\cos(t)}$ is positive and decreasing. Consequently, $e^{\cos(t)} > e^{\cos(a/2)}$ for $t < a/2$ and $e^{\cos(t)} < e^{\cos(a/2)}$ for $t > a/2$. Then

$$c(a) = \int_0^{a/2} e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt + \int_{a/2}^{a} e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt$$
$$> \int_0^{a/2} \min_{[0,a/2]} e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt + \int_{a/2}^{a} \max_{[a/2,a]} e^{\cos(t)} \sin\left(\frac{2\pi t}{a}\right) dt$$
$$= \int_0^{a/2} e^{\cos(a/2)} \sin\left(\frac{2\pi t}{a}\right) dt + \int_{a/2}^{a} e^{\cos(a/2)} \sin\left(\frac{2\pi t}{a}\right) dt = 0.$$

Therefore $c(a) > 0$ for $0 < a < \pi$. By [7], the Floquet multiplier of the impulse extension equation induced by (φ, a) is

$$\mu_a = X(2\pi)\left[1 + \int_0^a e^{\cos(t)-1} a^5 \sin\left(\frac{2\pi t}{a}\right) \sin\left(\frac{1}{a}\right) dt\right]$$
$$= 1 + \frac{1}{e} a^5 \sin\left(\frac{1}{a}\right) c(a).$$

The function $a^5 \sin\left(\frac{1}{a}\right)$ has roots at $(2\pi n)^{-1}$ for all integers n, with derivative oscillating in sign from positive to negative. Consequently, $a^5 \sin\left(\frac{1}{a}\right)$ assumes both positive and negative values on any interval $(0, \epsilon)$. We conclude that μ_a oscillates between greater than and less than 1 on any interval $(0, \epsilon)$ for $\epsilon < \pi$; see Fig. 1 for a visualization. In terms of stability, this means that the stability of the impulsive system (28) cannot be used to predict the stability of an associated impulse extension equation, even if the step sequence is very small.

In conclusion, the conditions of Corollary 3.1.4 on the spectrum of the impulsive monodromy matrix, M_0, cannot in general be weakened without assuming additional hypotheses on the family of impulse extensions.

4. The time-scale tolerance for linear, homogeneous impulsive differential equations

In this section, we introduce the notion of *uniformly* (σ, w)*-regulated families of impulse extensions* and the *time-scale tolerance* for linear, homogeneous impulsive differential equations. The definitions differ between periodic and aperiodic systems. Generally, we must deal with stable and unstable systems separately. First we have a basic definition. In this section, the word periodic will be synonymous with (T, c)-periodic.

Definition 4.1. Consider a homogeneous impulsive differential equation, (2). Let $\sigma = \{\sigma_k\}$ be a sequence (c-element, if (2) is periodic) of positive real numbers and $w = \{w_k\}$ be a sequence (c-element, if (2) is periodic) of functions $w_k : [\tau_k, \tau_{k+1}] \times \overline{\mathcal{S}_c^*} \to \mathbb{R}_+$ that are continuous and vanishing at $(\tau_k, 0)$ and such that $w_k(\cdot, a)$ is integrable on $\mathcal{S}_k(a)$. A family of periodic impulse extensions, $\varphi = \{\varphi_k\}$, is *uniformly exponentially* (σ, w)*-regulated in the mean* or simply *uniformly* (σ, w)*-regulated* if the inequality

$$\left|\left|\varphi_k(s, a) - \frac{1}{a_k} B_k\right|\right| \le \frac{w_k(s, a)}{e^{\sigma_k a_k} - 1} \tag{29}$$

is satisfied for all $s \in \mathcal{S}_k(a)$ and $k \in \mathbb{Z}$ (or $k = 0, \ldots, c-1$, of (2) is periodic). A pair (σ, w) that satisfies the above criteria will be referred to as a *uniform exponential regulator*. If φ is uniformly (σ, w)-regulated, we will write $\varphi \in (\sigma, w)$.

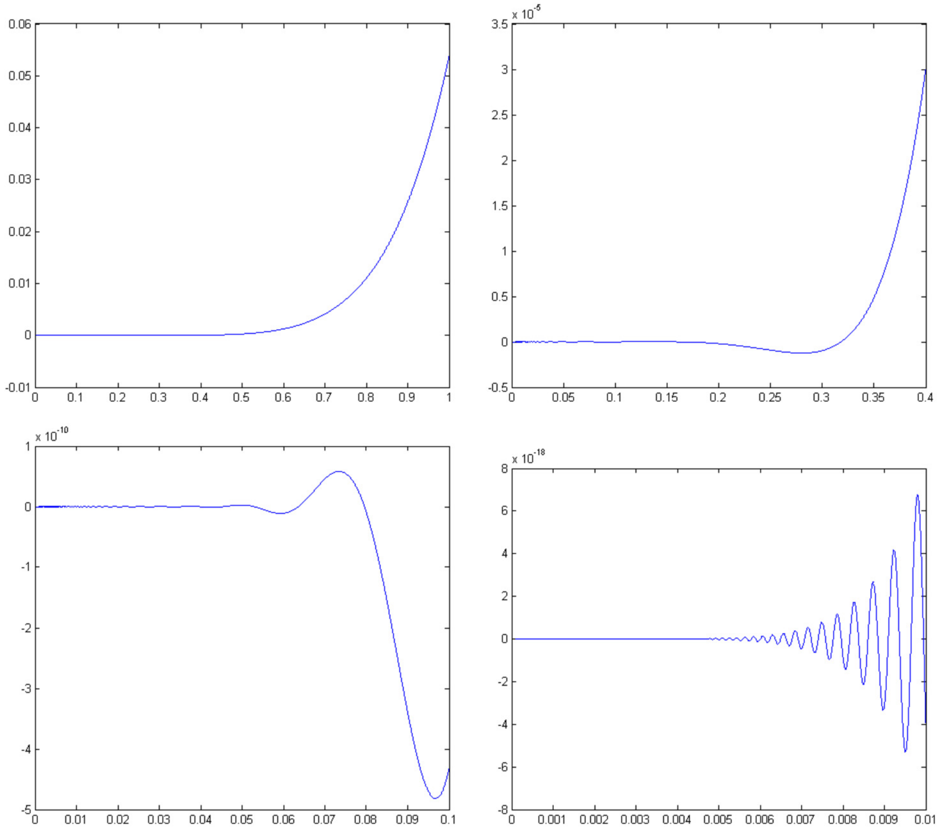

Fig. 1. Plots of $\mu_a - 1$ on four different scales, with 4000 sample points. Notice that oscillation is more easily seen on the smaller scales. This is to be expected, as the amplitude is essentially a fifth-order polynomial in a. This figure appears in [5].

Section 4.1 introduces the time-scale tolerance for asymptotically stable periodic systems, proving several elementary properties and providing an algorithm for its calculation. Section 4.2 extends the definition to unstable periodic systems. Finally, in Section 4.3, the definition of time-scale tolerance is extended to general, aperiodic systems via exponential dichotomies. For a physical interpretation of uniform exponential regulators, see Section 4.4.

4.1. The time-scale tolerance for periodic, asymptotically stable systems

We treat periodic, asymptotically stable homogeneous systems (2) first.

Definition 4.2. If $R = (\sigma, w)$ is a uniform exponential regulator and $a \in S_c^*$, the (R, a)-*pseudospectral radius of* (2), denoted $\rho(R, a)$, is defined by

$$\rho(R, a) = \sup_{\varphi \in R} \rho M(\varphi, a), \tag{30}$$

where $M(\varphi, a)$ is the monodromy matrix of the impulse extension equation for (2) induced by (φ, a).

Definition 4.3. Suppose (2) is asymptotically stable. Let R be a uniform exponential regulator. If φ is a periodic family of impulse extensions for (1), let $M(\varphi, a)$ denote the monodromy matrix of the impulse extension equation for (1) induced by (φ, a). The *R-stable set*, denoted $\mathcal{E}_s(R)$, is defined as follows.

$$\mathcal{E}_s(R) = \{ a \in S_c^* : \rho(R, a) < 1 \} . \tag{31}$$

The *R-time-scale tolerance* is the number

$$\mathcal{E}_t(R) = \sup\{\epsilon : \exists a \in \mathcal{E}_s(R), ||a|| = \epsilon, B_\epsilon(0) \cap S_c^* \subseteq \mathcal{E}_s(R)\}. \tag{32}$$

The time-scale tolerance is defined precisely so that we have the following elementary property, whose proof we omit.

Proposition 4.1. *Given a uniform exponential regulator $R = (\sigma, w)$, the time-scale tolerance behaves as a robust stability threshold for the impulsive system (1); if $||a|| < \mathcal{E}_t(R)$, then $\rho(R, a) < 1$. In other words, systems (1) the impulse extension equation (5) induced by (φ, a) are both stable, for all $\varphi \in R$.*

As should be expected, if the regulator is not chosen wisely, the time-scale tolerance for the given regulator might be zero. This is not the case if one obeys the guidelines of, for example, Corollary 3.1.3.

Theorem 4.1. *Suppose $\sigma \in \{\sigma_A, \sigma_F\}$. If $R_\sigma = (\sigma, w)$ is a uniform exponential regulator and (2) is asymptotically stable (i.e. $\rho M_0 < 1$), then $\mathcal{E}_t(R_\sigma)$ is nonzero and the map $a \mapsto \rho(R_\sigma, a)$ satisfies*

$$\lim_{a\to 0} \rho(R_\sigma, a) = \rho M_0,$$

where the limit is for $a \in S_c^*$.

Proof. By the proof of Lemma 3.1, the bound

$$\left| \int\limits_{\mathcal{S}_k(a)} X^{-1}(s;\tau_k)\epsilon_k(s;a)ds \right| \leq \frac{||w_k(s,a)||_\infty^{\mathcal{S}_k(a)}}{\sigma}$$

holds for $k = 0, \dots, c-1$, for all $\varphi_k = \frac{1}{a_k} + \epsilon_k \in (\sigma, w)$. That is, the bound is independent of the choice of φ. Since we can write

$$M(\varphi, a) = \prod_{k=c-1}^{0} X(\tau_{k+1};\tau_k) \left(\int\limits_{\mathcal{S}_k(a)} X^{-1}(s;\tau_k)\epsilon_k(s,a)ds + \frac{1}{a_k} \int\limits_{\mathcal{S}_k(a)} I + X^{-1}(s;\tau_k)B_k ds \right),$$

it follows that $M(\varphi, a)$ converges to M_0 as $||a|| \to 0$, uniformly in φ. Consequently, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for $||a|| < \delta$, we have $||M(\varphi, a) - M_0|| < \epsilon$ for all $\varphi \in (\sigma, w)$. From the continuity of the spectral radius map, $X \mapsto \rho X$, we conclude that, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for $||a|| < \delta$, we have $|\rho M(\varphi, a) - \rho M_0| < \epsilon$ for all $\varphi \in (\sigma, w)$. In particular, we must have

$$\left| \sup_{\varphi \in (\sigma, w)} \rho M(\varphi, a) - M_0 \right| = |\rho(R_\sigma, a) - \rho M_0| < \epsilon.$$

We conclude that $\rho(R_\sigma, a) \to \rho M_0$ as $a \to 0$ in S_c^*.

If we choose $\epsilon = |1 - \rho M_0|$, there exists $\delta > 0$ such that

$$|\rho(R_\sigma, a) - \rho M_0| < |1 - \rho(M_0)|,$$

provided $||a|| < \delta$. Consequently, $\rho(R_\sigma, a) < 1$ for this range of $||a|| < \delta$, indicating that the set $\mathcal{E}_s(R_\sigma)$ contains $B_\delta(0) \cap \mathcal{S}_c^*$. It follows that the set

$$\{\epsilon : \exists a \in \mathcal{E}_s(R_\sigma) : |a| = \epsilon, B_\epsilon(0) \cap \mathcal{S}_c^* \subseteq \mathcal{E}_s(R_\sigma)\}$$

is nonempty and contains $\delta > 0$ and therefore has nonzero least upper bound. This least upper bound is precisely the time-scale tolerance, $\mathcal{E}_t(R_\sigma)$. □

The function $a \mapsto \rho(R_\sigma, a)$ can be made continuous on the entirety of $\mathcal{S}_c^*$, although the most natural assumption to impose requires restricting to sets of uniformly (σ, w)-regulated families of impulse extensions that also satisfy an equicontinuity-like condition. Such an assumption is too strong to impose for most practical problems; as such, the result is not very helpful and is omitted.

In practice, the time-scale tolerance is difficult to calculate. It is much easier to provide a method of finding a lower bound to the time-scale tolerance by taking advantage of its definition, which allows for approximation by pseudospectral radii. Recall that the ϵ-pseudospectral radius, $\rho_\epsilon A$, of a matrix A is defined by

$$\rho_\epsilon A = \max\{\rho B : ||A - B|| \leq \epsilon\}. \tag{33}$$

For additional information about the pseudospectral radius, other pseudospectra and their computation, see [9,11,13]. In particular, we have the following proposition.

Proposition 4.2. *Let a uniform exponential regulator R for the (T, c)-periodic impulsive differential equation (2) be given. Suppose the inequality*

$$||M(\varphi, a) - M_0|| \leq n(a) \tag{34}$$

is satisfied for all $\varphi \in R$ and all $a \in S_c^*$, for some function $n : \mathcal{S}_c^* \to \mathbb{R}$. The following are true.

1. $\rho(R, a) \leq \rho_{n(a)} M_0$ *for all* $a \in S_c^*$.
2. *The following inclusion is valid:*

$$\widehat{\mathcal{E}_s}(R) \equiv \{a \in S_c^* : \rho_{n(a)} M_0 < 1\} \subseteq \mathcal{E}_s(R).$$

3. *Let h denote the unique solution of the equation $\rho_h M_0 = 1$. The inequality*

$$\widehat{\mathcal{E}_t}(R) \equiv \min\{||a|| : n(a) = h, a \in \overline{S_c^*}\} \leq \mathcal{E}_t(R)$$

is valid. If $||a|| < \widehat{\mathcal{E}_t}(R)$, then $\rho M(\varphi, a) < 1$ for all $\varphi \in R$.

Proof. By definition of the pseudospectral radius, we have

$$\begin{aligned}\rho_{n(a)} M_0 &= \sup\{\rho(Z) : Z \in \mathbb{R}^{n \times n}, ||Z - M_0|| \leq n(a)\}\\ &\geq \sup\{\rho M(\varphi, a) : \varphi \in R, \; ||M(\varphi, a) - M_0|| \leq n(a)\}\\ &= \sup\{\rho M(\varphi, a) : \varphi \in R\} = \rho(R, a),\end{aligned}$$

where the inequality follows by condition (34). The other two conclusions of the theorem follow directly from the above inequality. □

Construction of an appropriate function $n : \mathcal{S}_c^* \to \mathbb{R}$ that satisfies the condition of inequality (34) is important. Additional desirable properties include having n be continuous and strictly monotone increasing, for then the set $\widehat{\mathcal{E}_s}(R)$ becomes star-convex and $\widehat{\mathcal{E}_t}(R)$ can be seen as the minimum norm of all vectors lying in a compact hypersurface $(c - 1)$-dimensional hypersurface.

Lemma 4.1. *Let a uniform exponential regulator R be given, and suppose there exists a continuous, monotone nondecreasing function $n : \mathcal{S}_c^* \to \mathbb{R}$ satisfying inequality (34). Then $\widehat{\mathcal{E}}_s(R)$ is star-convex with basepoint $0 \in \overline{\mathcal{S}_c^*}$. If, in addition, n is strictly monotone increasing and extends to a continuous function $\overline{n} : \overline{\mathcal{S}_c^*} \to \mathbb{R}$ and $\mathcal{E}_t(R) < \infty$, then the set*

$$\widehat{\mathcal{E}}_s^+(R) = \{a \in \overline{\mathcal{S}_c^*} : \overline{n}(a) = h\} \tag{35}$$

is a compact hypersurface and $\widehat{\mathcal{E}}_t(R) = \min\{||a|| : a \in \widehat{\mathcal{E}}_s^+(R)\}$, where h denotes the unique solution of the equation $\rho_h M_0 = 1$.

Proof. For simplicity of notation, we will write $U = \widehat{\mathcal{E}}_s(R)$. We first prove U is star-convex with basepoint 0. If $a \in U$, it follows that, for all $t \in (0, 1)$, at least one index of ta must be strictly less than the corresponding index of a; for example, suppose $(ta)_k < a_k$. Since n is monotone nondecreasing, we obtain $n(ta) \leq n(a)$, which implies, by the monotonicity of the pseudospectral radius, that $\rho_{n(ta)} M_0 \leq \rho_{n(a)} M_0 < 1$. Star-convexity of U follows.

Next we show that $V = \widehat{\mathcal{E}}_s^+(R)$ is a compact hypersurface. Define a map $\Psi : V \to \Psi(V) \subset \mathbb{R}^{c-1}$ by $V(x^1, x^2, \ldots, x^c) = (x^1, x^2, \ldots, x^{c-1})$. Since Ψ is a projection, it is continuous.

Now let $y \in \Psi(V)$. Since $y \in \Psi(V)$, there exists $y^c \in [0, \Delta\tau_{c-1}]$ such that $n(y, y^c) = h$. However, since n is strictly monotone increasing, we can have $n(y, y^c) = h = n(y, t)$ if and only if $y^c = t$. Consequently, to each $y \in \Psi(V)$, we can associate a unique $y^c \in [0, \Delta\tau_{c-1}]$ such that $(y, y^c) \in V$. It follows that Ψ is invertible and $\Psi^{-1}(y) = (y, y^c)$.

Next we show that Ψ^{-1} is continuous. Suppose Ψ^{-1} is discontinuous at some $y \in \Psi(V)$, so there exists some sequence $y_n \to y$ with $\Psi^{-1}(y_n) \nrightarrow \Psi^{-1}(y)$. Since V is compact, it follows that there exists a subsequence, also denoted y_n, such that $\Psi^{-1}(y_n) \to x \neq \Psi^{-1}(y)$. By compactness, $x \in V$, so there must be some $z \in \Psi(V)$ such that $x = \Psi^{-1}(z)$. Hence $\Psi^{-1}(y_n) \to \Psi^{-1}(z)$. By continuity of Ψ, we have

$$y_n = \Psi(\Psi^{-1}(y_n)) \to \Psi(\Psi^{-1}(z)) = z.$$

By uniqueness of limits, since $y_n \to y$ and $y_n \to z$, we must have $y = z$. Therefore $\Psi^{-1}(y) = \Psi^{-1}(z) = x$, which is a contradiction to $\Psi^{-1}(y) \neq x$. We conclude that Ψ^{-1} is continuous and hence that Ψ is a homeomorphism. Therefore V is a compact hypersurface. □

In finding a function n satisfying inequality (34), the following combinatorial representation of $||M(\varphi, a) - M_0||$ is helpful. The proof follows by an inductive argument and is omitted.

Lemma 4.2. *For natural number $z \leq c - 1$, let Θ_z denote the $\binom{c}{z}$-element sequence of z-element subsets of the set $\{0, 1, \ldots, c-1\}$, let $\Theta_z(n)$ denote the nth element*[1] *of this sequence and let $\overline{\Theta}_z(n)$ denote its complement in $\{0, 1, \ldots, c-1\}$. For all periodic impulse extensions $\varphi = \{\varphi_k\}$ for the periodic impulsive differential equation (2), we have the inequality*

$$||M(\varphi, a) - M_0|| \leq \sum_{k=0}^{c-1} \sum_{r=1}^{\binom{c}{k}} \left[\prod_{j \in \Theta_k(r)} ||X(\tau_{j+1}; \tau_j)(E + B_j)|| \prod_{v \in \overline{\Theta}_k(r)} ||X(\tau_{v+1}; \tau_v) [C_v + P_v]|| \right], \tag{36}$$

where $$C_v = C_v(\varphi, a) = \frac{1}{a_v} \left[\int_{\mathcal{S}_v(a)} (X^{-1}(s; \tau_v) - I) ds \right] B_v, \tag{37}$$

[1] For a consistent ordering, note that it is always possible to uniquely order the sequence Θ_z in such a way that the nth element, $\Theta_z(n)$, satisfies $\sum_{x \in \Theta_z(n)} x = n - 1 + \frac{z(z-1)}{2}$.

$$P_v = P_v(\varphi, a) = \int\limits_{\mathcal{S}_v(a)} X^{-1}(s;\tau_v)\left[\varphi_v(s,a) - \frac{1}{a_v}B_v\right] ds. \tag{38}$$

The bound appearing in (36) generally increases exponentially with c, since there are $2^c - 1$ terms in the sum. Moreover, the bound is not optimal, since it is obtained by repeated application of the triangle inequality. It is always possible to express $M(\varphi, a) - M_0$ exactly; that is, without resorting to upper bounds. The following examples provide exact formulae for small values of c.

Example 4.2.1. If $c = 1$, we have the equality

$$M(\varphi, a) - M_0 = X(T + \tau_0; \tau_0)\left[C_0 + P_0\right]. \tag{39}$$

Example 4.2.2. If $c = 2$, we have the equality

$$M(\varphi, a) - M_0 = X(\tau_2;\tau_1)\Big[[C_1 + P_1]\Big(X(\tau_1;\tau_0)\big[C_0 + P_0 + I + B_0\big]\Big) + (I + B_1)X(\tau_1;\tau_0)[C_0 + P_0]\Big] \tag{40}$$

The bound appearing in (36) indicates that, to ensure the existence of an upper bound of the form (34) that is continuous (and possibly monotone increasing) and defined on the closure of $\mathcal{S}_c^*$, it is enough to ensure that each of the functions C_v and P_v of (36)–(37) each have continuous (and possibly monotone increasing) upper bounds with respect to the input $a \in \overline{\mathcal{S}_c^*}$, where the bound holds uniformly for all $\varphi \in R$, given a regulator R.

In the following, we outline an algorithm that can be used to compute $\widehat{\mathcal{E}}_t(R)$. To this end, we write the symbolic expression appearing on the right-hand side of (36) as a function of the functions C_k and P_k. The proof follows from the lemmas of this section and the above remark and is omitted.

$$n(a, C, P) = \sum_{k=0}^{c-1}\sum_{r=1}^{\binom{c}{k}}\left[\prod_{j\in\Theta_k(r)} ||X(\tau_{j+1};\tau_j)(E + B_j)|| \prod_{v\in\overline{\Theta}_k(r)} ||X(\tau_{v+1};\tau_v)\left[C_v + P_v\right]||\right]. \tag{41}$$

Algorithm 4.1. Let $R = (\sigma, w)$ be a uniform exponential regulator for the asymptotically stable (T, c)-periodic impulsive differential equation (2). Suppose σ is defined as in (17) and w satisfies the conditions of Lemma 4.3.

1. Choose continuous (and possibly monotone increasing) functions $C^+ : \overline{\mathcal{S}_c^*} \to \mathbb{R}_+^c$ and $P^+ : \overline{\mathcal{S}_c^*} \to \mathbb{R}_+^c$ that satisfy the inequalities $||C_k(a)|| \le C_k^+(a)$ and $||P_k(a)|| \le P_k^+(a)$ for $k = 0, \ldots, c-1$.
2. Calculate the unique solution, $h > 0$, of the equation $\rho_h M_0 = 1$.
3. Find the global minimizer, a^*, of the function $f(a) = ||a||$, subject to the constraints $a \in \overline{S_c^*}$ and $n(a, C^+, P^+) - h = 0$.

Then, $||a^*|| = \widehat{\mathcal{E}}_t(R)$.

4.1.1. Choices of C^+ and P^+ guaranteeing monotonicity of $n(a, C^+, P^+)$

Under certain assumptions on the uniform exponential regulator, we can guarantee the existence of monotone increasing upper bounds for C and P.

Lemma 4.3. *Let $R = (\sigma, w)$ be an exponential regulator for the (T, c)-periodic equation (2), with σ as given in (17). Suppose the functions $a \mapsto \sup_{s\in\mathcal{S}_k(a)} ||w_k(s, a)||$ are continuous for each $k = 0, \ldots, c-1$. There exist functions C_k^+ and P_k^+, with $k = 0, \ldots, c-1$, mapping $\overline{\mathcal{S}_c^*} \to \mathbb{R}^+$, satisfying the inequalities*

$||C_k(\varphi, a)|| \leq C_k^+(a)$ and $||P_k(\varphi, a)|| \leq P_k^+(a)$ for all $a \in S_c^*$ and all $\varphi \in R$. The functions C_k^+ and P_k^+ are monotone nondecreasing on $\mathcal{S}_c^*$, and if $a < b$ with $a_k < b_k$, then $C_k^+(a) < C_k^+(b)$ and $P_k^+(a) < P_k^+(b)$.

Proof. We define the functions C_k^+ and P_k^+ as follows.

$$\begin{aligned} C_k^+(a; \sup) &= \sup_{s \in \mathcal{S}_k(a)} ||(X^{-1}(s; \tau_k) - I)B_k|| \\ P_k^+(a; \sup) &= \sup_{s \in \mathcal{S}_k(a)} w_k(s, a). \end{aligned} \tag{42}$$

C_k^+ is clearly monotone nondecreasing, as it is defined by the supremum of a continuous function on the set $\mathcal{S}_k(a)$, which satisfies the inclusion $\mathcal{S}_k(a) \subseteq \mathcal{S}_k(b)$ whenever $a \leq b$. The situation is similar for $P_k^+(a)$, due to the hypotheses on the functions w_k. It also follows that $P_k^+(a) < P_k^+(b)$ whenever $a < b$ and $a_k < b_k$. The inequalities $||C_k(a)|| \leq C_k^+(a)$ and $||P_k(a)|| \leq P_k^+(a)$ follow from elementary integral inequalities and inequality (29). □

Note that the functions C_k^+ and P_k^+ described in the proof of Lemma 4.3 may not be optimal, in that there may be uniform bounds for C_k and P_k that are monotone increasing but smaller than the bounds provided by the lemma. For example, the following bounds hold uniformly for $\varphi \in (\sigma, w)$:

$$C_k \leq \frac{1}{a_k} \int_{\mathcal{S}_k(a)} ||(X^{-1}(s; \tau_k) - I)B_k|| ds \equiv C_k^+(A; \text{Int}), \tag{43}$$

$$P_k \leq \int_{\mathcal{S}_k(a)} \frac{e^{\sigma(\tau_k - s)} w_k(s, a)}{e^{\sigma a_k} - 1} ds \equiv P_k^+(a; \text{Int}), \tag{44}$$

$$P_k \leq \sqrt{\frac{1}{2\sigma} \cdot \frac{e^{2\sigma a_k} - 1}{(e^{\sigma a_k} - 1)^2} \int_{\mathcal{S}_k(a)} w_k^2(s, a) ds} \equiv P_k^+(a; \text{CS}). \tag{45}$$

Depending on the specific application, we could be more conservative. If the upper bounds are still monotone increasing in a_k and continuous in a, they could be more suitable for the purposes of approximating the time-scale tolerance.

4.1.2. Discussion of Algorithm 4.1

In practice, implementing steps 1 and 2 Algorithm 4.1 do not pose much difficulty. Step 1 always has a worst-case choice to fall back on: $C_k^+(a; \sup)$ and $P_k^+(a; \sup)$. The bounds $P_k^+(a; \text{Int})$ and $P_k^+(a; \text{CS})$ of (44)–(45) could be computed exactly for specific choices of uniform regulators $R = (\sigma, w)$. There is also the upper bound for C_k provided by $C_k^+(a; \text{Int})$ of (43). Note that all of these bounds can be ensured to be continuous (even if they are not monotone increasing) by an appropriate choice of uniform exponential regulator. If one wishes for the bounds to be monotone increasing, it is worth mentioning that all of the suggested bounds for P can be made monotone increasing by an appropriate choice of exponential regulator, and the monotonicity of the bounds for C could be tested statistically, if needed.

We can also choose an optimal bound by simply taking the minimum of any particular set of bounds. For example, if one chooses

$$\begin{aligned} P_k^+(a) &= \min\{P_k^+(a; \sup), P_k^+(a; \text{Int}), P_k^+(a; \text{CS})\}, \\ C_k^+(a) &= \min\{C_k^+(a; \sup), C_k^+(a; \text{Int})\}, \end{aligned} \tag{46}$$

the resulting functions P^+ and C^+ will be continuous (they are finite minimums of continuous functions) and increasing, provided each estimate is also increasing (since a minimum of increasing functions is increasing). By construction, they provide tighter estimates than each individual bound.

The second step of the algorithm involves solving the equation $\rho_h M_0 = 1$ for $h > 0$. Since $h \in \mathbb{R}$ and $h \mapsto \rho_h M_0$ is monotone nondecreasing (but typically nonsmooth), the bisection method is applicable.

The third step will typically be the most computationally expensive. While $\overline{S_c^*}$ is convex and the objective $f(a) = ||a||$ is convex, the other constraint, $c(a) = 0$ with $c(a) = n(a, C^+, P^+) - h$, generally destroys the convexity of the domain; the resulting set, $\widehat{\mathcal{E}}_s(R)$, could have no "nice" structure, or it could be star-convex, by Lemma 4.1. We comment on a few methods now.

Monotonic optimization by reverse polyblock approximation Suppose $n(a) = n(a, C^P, P^+)$ is monotone. The objective, $a \mapsto ||a||$, is also monotone, and the domain, $\overline{S_c^*}$, is convex. This problem can therefore be solved by reverse polyblock approximation as follows. Following [20], define $G \equiv \overline{S_c^*} \subset [0, b]$, with $b = \max \Delta\tau_k$; G is compact and normal with nonempty interior. If we take $H \equiv \mathbb{R}^n_+ \setminus \widehat{\mathcal{E}}_s(R)^\circ$, then H is closed, and its complement in $\mathbb{R}^n_+$ is $\widehat{\mathcal{E}}_s(R)^\circ$, which is a normal set since $\widehat{\mathcal{E}}_s(R)^\circ$ is defined by $0 \le n(a) < h$ and n is increasing. Therefore H is closed and reverse normal. By construction, $G \cap H$ contains the level set $\{a \in \overline{S_c^*} : n(a) = h\} = \widehat{\mathcal{E}}_s^+(R)$.

Now define the objective function $f : [0, b] \to \mathbb{R}_+$ by $f(a) = ||a||$. By Proposition 11 of [20], any minimizer of the problem

$$\min\{f(a) : a \in G \cap H\} \tag{47}$$

must be an element of $\partial^- H = \widehat{\mathcal{E}}_s^+(R)$. Consequently, a global minimizer a^* of (47) satisfies $n(a^*) - h = 0$ and minimizes $a \mapsto ||a||$ over the level set $\widehat{\mathcal{E}}_s^+(R)$. By Proposition 4.2, a global minimizer a^* of problem (47) satisfies $||a^*|| = \widehat{\mathcal{E}}_t(R)$. The reverse polyblock approximation algorithm, described in [20], finds an ϵ-optimal solution, which, for our problem, means that the approximate minimizer $\overline{a}^*$ satisfies the inequality

$$\widehat{\mathcal{E}}_t(R, \epsilon) \equiv ||\overline{a}^*|| - \epsilon \le \widehat{\mathcal{E}}_t(R).$$

However, since $\overline{a}^*$ is a feasible solution, we must have $||\overline{a}^*|| \ge \widehat{\mathcal{E}}_t(R)$. Using this fact and rearranging the above inequality, we obtain

$$0 \le \widehat{\mathcal{E}}_t(R) - \widehat{\mathcal{E}}_t(R, \epsilon) \le \epsilon. \tag{48}$$

Therefore the reverse polyblock approximation algorithm generates an ϵ-underestimate of $\widehat{\mathcal{E}}_t(R)$, which we call $\widehat{\mathcal{E}}_t(R, \epsilon)$.

Lower approximation by piecewise-constant functions on a grid When $c = 1$ and $n(a)$ is continuous and monotone strictly increasing, the problem is trivial to solve, since all that is needed is to solve the equation $n(a) = h$ for scalar $a \in [0, \Delta\tau_0]$. This can be accomplished by the bisection method, or possibly a quasi-Newton method. Moreover, there is a unique solution when n is monotone strictly increasing. If n is only monotone nondecreasing, a quasi-Newton method to find a feasible solution followed by a some sort of bracketing method should be sufficient to bracket the minimal solution to any desired level of precision.

When $c = 2$ and $n(a)$ is continuous and strictly monotone increasing, the hypersurface $\widehat{\mathcal{E}}_s^+(R)$ is one-dimensional. If $[0, \Delta\tau_0]$ is discretized into a grid with N cells, $[0, a_0^1], [a_0^1, a_0^2], \ldots, [a_0^{N-1}, a_0^N]$, $\widehat{\mathcal{E}}_s^+(R)$ can be parameterized along the vertices of the cells by solving the equation $n(a_0^m, a_1^m) = h$ for $a_1^m \in [0, \Delta\tau_1]$,[2] for

[2] If no solution exists, set $a_1^m = \Delta\tau_1$.

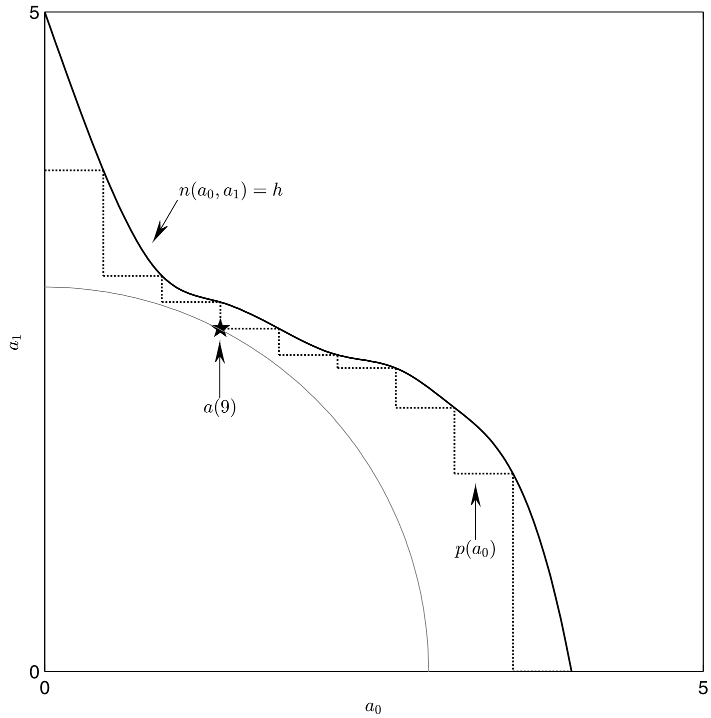

Fig. 2. Plot of a (theoretical, for illustrative purposes only) hypersurface, $n(a) = h$, with n continuous and monotone strictly increasing, and $c = 2$ impulses per period. The piecewise-constant under-approximation, $p(a_0)$, generated by a grid with 9 cells, is plotted (thin dotted black line), and the point that generates the lower estimate for the time-scale tolerance is indicated by a star. All points $a = (a_0, a_1)$ within the interior of the disc (grey line) with radius $r = ||a(9)|| < \widehat{\mathcal{E}}_t(R)$ (notice that the inequality is strict because the disc does not intersect the hypersurface) would satisfy the inequality $\rho(R, a) < 1$. Also, one can see that the upper bound provided by (49)–(50) is not very conservative in this case; the bound can certainly be improved, although the notation gets cumbersome.

each vertex a_0^m. A piecewise-constant under-approximation of the parameterization can then be constructed as follows.

$$p(a_0) = \begin{cases} a_1^{m+1}, & a_0 \in [a_0^m, a_0^{m+1}), \\ a_1^N, & a_0 = a_0^N. \end{cases}$$

The function p is indeed an under-approximation, since, for $a_0 \in [a_0^m, a_0^{m+1})$, we have $n(a_0, p(a_0)) \leq n(a_0^{m+1}, a_1^{m+1}) = h$, and $n(a_0^N, p(a_0^N)) = h$. If one calculates

$$a(N) = \text{argmin}\{||a|| : a = (a_0^m, a_1^{m+1}) : m = 1, \dots, N-1\},$$

then, by construction, $||a(N)|| \leq \widehat{\mathcal{E}}_t(R)$. In particular, one can show that the inequality

$$0 \leq \widehat{\mathcal{E}}_t(R) - ||a(N)|| \leq \left(\frac{\Delta\tau_1^2}{N^2} + \max |\Delta a_1^m|^2 \right)^{\frac{1}{2}} \tag{49}$$

holds. Since n is continuous, the maximum term becomes arbitrarily small as $N \to \infty$. Therefore, to obtain the precision desired, one needs only iterate the procedure on N, successively subdividing intervals, until the right-hand side is smaller than the desired tolerance. See Fig. 2 for a visualization.

The above approach can be similarly applied to problems with cycle number $c > 2$, with slight modifications. If $C = [x_0, y_0] \times \cdots \times [x_{c-1}, y_{c-1}] \subset \mathbb{R}^c$ is a cell, we denote $C^- = [x_0, y_0) \times \cdots \times [x_{c-1}, y_{c-1})$ and

$$C^l = (x_0, \dots, x_{c-1}), \qquad C^r = (y_0, \dots, y_{c-1}).$$

The modification is that $[0, \Delta\tau_0] \times [0, \Delta\tau_{c-1}]$ is discretized into cells C_m, $m = 1, \dots, N^{2^{c-1}}$, and the function p is defined in such a way that

$$p(C_m^-) = \arg\{a_{c-1} : n(C_m^r, a_{c-1}) = h\}.$$

The rest of the algorithm is essentially unchanged; $a(N)$ is the argument that minimizes $||a||$ over the set of $a = (C_m^l, p(C_m^-))$. The resulting bound satisfies the inequality

$$0 \leq \widetilde{\mathcal{E}}_t(R) - ||a(N)|| \leq \left(\frac{(c-1)\max \Delta\tau_i^2}{N^2} + \max |p(C_m^-) - p(\pi C_m^-)|^2 \right)^{\frac{1}{2}}, \tag{50}$$

where π is a partial function on half-open cells that maps a given cell to the one that is upper diagonal to it; the map is defined by the equivalence

$$\pi(C_m^-) = C_j^- \iff C_m^r = C_j^l.$$

Note that the maximum is only taken over those cells where $\pi(C_m^-)$ exists (these are the cells for which C_m^r is not an element of the boundary of $[0, \Delta\tau_0] \times \dots [\Delta\tau_{c-1}]$). Again, the above can be iterated, taking N as large as needed, since the maximum term consists of a difference between evaluations of a continuous function defined at opposing vertices of a hypercube of side length $\frac{1}{N}$, which will become arbitrarily small as $N \to 0$. The iterations require more recursion than in the case $c = 2$, however.

4.2. The time-scale tolerance for unstable periodic systems

The time-scale tolerance can be defined for unstable systems as well, provided certain conditions on the center subspace of the iterated map $x \mapsto M_0 x$ hold. If there is a center subspace, it is possible for the spectral radius to oscillate between greater than or less than one on any time scale, as Example 3.4.2 demonstrates. This defect makes it generally impossible to study time-scale tolerances in systems for which there is a center subspace but no unstable subspace. However, if there is an unstable subspace, such defects do not cause issues. The analysis of this section is inspired by a short discussion appearing in [4].

Definition 4.4. If $R = (\sigma, w)$ is a uniform exponential regulator and $a \in S_c^*$, the (R, a)-*lower pseudospectral radius of* (2), denoted $\rho^-(R, a)$, is defined by

$$\rho^-(R, a) = \inf_{\varphi \in R} \rho M(\varphi, a). \tag{51}$$

The following proposition appears in [4].

Proposition 4.3. *Let R be a uniform exponential regulator for* (2). *Suppose $||M(\varphi, a) - M_0|| \leq n(a)$ for some continuous function $n(a)$ satisfying $n(0) = 0$, for all $a \in S_c^*$. The following inequality holds.*

$$\rho^-(R, a) \geq \rho_{n(a)}^- M_0 \equiv \inf\{\rho M : ||M - M_0|| \leq n(a)\}. \tag{52}$$

Proof. We follow the string of inequalities

$$\begin{aligned}\inf_{\varphi \in R} \rho M(\varphi, a) &\geq \inf \left\{ \rho M : ||M - M_0|| \leq \sup_{\varphi \in R} ||M(\varphi, a) - M_0|| \right\} \\ &= \inf\{\rho M : ||M - M_0|| \leq \inf\{x : ||M(\varphi, a) - M_0|| \leq x, \forall \varphi \in R\}\} \\ &\geq \inf\{\rho M : ||M - M_0|| \leq n(a)\} = \rho_{n(a)}^- M_0,\end{aligned}$$

thereby obtaining the result claimed. □

Definition 4.5. Suppose the (T, c)-periodic impulsive system (2) has no Floquet multipliers on the unit circle and is unstable. If R is a uniform exponential regulator, the *R-unstable set*, denoted $\mathcal{E}_u(R)$, is defined as follows.

$$\mathcal{E}_u(R) = \left\{a \in S_c^* : \rho^-(R, a) > 1\right\}. \tag{53}$$

The *R-time-scale tolerance* is the number

$$\mathcal{E}_t(R) = \sup\{\epsilon : \exists a \in \mathcal{E}_u(R), ||a|| = \epsilon, B_\epsilon(0) \cap S_c^* \subseteq \mathcal{E}_u(R)\}; \tag{54}$$

the time-scale tolerance is defined as for stable systems.

The proof of the following proposition is essentially the same as the analogous proof of Theorem 4.1, and is omitted.

Proposition 4.4. *Let M_0 denote the monodromy matrix for the (T, c)-periodic equation (2). Let $R = (\sigma, w)$ be a uniform exponential regulator with $\sigma \in \{\sigma_A, \sigma_F\}$. Suppose $\rho M_0 > 1$.*

1. *The R-time-scale tolerance exists.*
2. $\lim_{a\to 0} \rho^-(R, a) = \rho M_0$, *where the limit is for $a \in S_c^*$.*

Once again, the time-scale tolerance behaves as a robust (in)stability threshold. If $||a|| < \mathcal{E}_t(R)$, then the impulse extension equation induced by (φ, a) will be unstable for all $\varphi \in R$. The following corollary is obvious and is not proven.

Corollary 4.1.1. *Suppose the conditions of Proposition 4.3 hold. Let $\widehat{\mathcal{E}}_t(R)$ be the solution of the optimization problem*

$$\widehat{\mathcal{E}}_t(R) \equiv \sup\{||a|| : a \in B_{||a||}(0) \subset \widehat{\mathcal{E}}_u(R)\}, \tag{55}$$

with

$$\widehat{\mathcal{E}}_u(R) = \{a \in S_c^* : \rho^-_{n(a)} M_0 > 1\}.$$

Then $\mathcal{E}_t(R) \geq \widehat{\mathcal{E}}_t(R)$, where $\mathcal{E}_t(R)$ is the R-time-scale tolerance for the (T, c)-periodic impulsive system (2) satisfying $\rho M_0 > 1$. If n is monotone strictly increasing and extends continuously to $\overline{S_c^*}$, then

$$\widehat{\mathcal{E}}_t(R) = \min\{||a|| : \rho^-_{n(a)} M_0 = 1,\ a \in \overline{S_c^*}\}. \tag{56}$$

The above problem is not as well-posed as the associated problem for asymptotically stable systems because the map

$$M \mapsto \rho^-_\epsilon M = \min\{\rho N : ||N - M|| \leq \epsilon\}$$

is not as well-behaved from a numerical perspective, and the computation of this map is an essential step in calculating $\widehat{\mathcal{E}}_t(R)$ as in (56). For background on the problem of minimizing the spectral radius, one should consult the works of, for example, Burke, Lewis and Overton [3], Overton and Womersley [16] and Nesterov and Protasov [15]. For our purposes, however, it is not difficult to see that the map $\epsilon \mapsto \rho^-_\epsilon M$ is monotone decreasing (although not strictly decreasing, since $\rho^-_\epsilon M = 0$ for $\epsilon \geq ||M||$, for example) and continuous for each fixed M, so the composition $a \mapsto \rho^-_{n(a)} M_0$ will generally be continuous and monotone decreasing, provided n is continuous and increasing. As such, assuming $\rho^{-}_{n(a)}M_0$ can be computed, Algorithm 4.1 and subsequent discussions can be adapted to the present case of unstable impulsive systems. We will not delve further into the problem at this time.

4.3. The time-scale tolerance for general homogeneous linear systems

If (2) is not periodic, one can abstractly define the time-scale tolerance via exponential dichotomies. For brevity, in this section, the symbol $E(\varphi, a)$ will refer to the impulse extension equation for (2) induced by (φ, a). In this section, without loss of generality, we take $\tau_0 = 0$.

Definition 4.6. The impulse extension equation $E(\varphi, a)$ possesses an *exponential dichotomy* if there exists a projector P such that the fundamental matrix solution of $E(\varphi, a)$, denoted $U(t)$ and satisfying $U(0) = I$, satisfies the inequalities

$$||U(t)PU^{-1}(s)|| \le Ke^{-\alpha(t-s)} \qquad s \le t < \infty \tag{57}$$

$$||U(t)(I-P)U^{-1}(s)|| \le Le^{-\beta(s-t)} \qquad s \ge t > -\infty \tag{58}$$

for positive constants α, β, K, L, whenever $s \in \mathcal{P}(\varphi, a)$. In this case, we will write $E(\varphi, a) \sim P$.

Definition 4.7. Suppose (2) possesses an exponential dichotomy with projector P_0. Let a uniform exponential regulator $R = (\sigma, w)$ for (2) be given. The *R-stable* and *R-unstable sets* are defined as follows.

$$\mathcal{E}_s(R) = \{a \in S^* : \forall \varphi \in R,\ \exists P : E(\varphi, a) \sim P,\ rank(P) = n\} \tag{59}$$

$$\mathcal{E}_u(R) = \{a \in S^* : \forall \varphi \in R,\ \exists P : E(\varphi, a) \sim P,\ rank(I-P) \ge 1\}. \tag{60}$$

By construction, $\mathcal{E}_s(R)$ and $\mathcal{E}_u(R)$ are disjoint.

Definition 4.8. Let R be a uniform exponential regulator for (2). The *R-time-scale tolerance* is the number

$$\mathcal{E}_t(R) = \sup\{||a|| : a \in B_{||a||}(0) \cap \mathcal{S}^* \subseteq \mathcal{E}_s(R) \vee \mathcal{E}_u(R)\} \tag{61}$$

provided it is positive, where the notation $X \subseteq Y \vee Z$ is understood as $X \subseteq Y \vee X \subseteq Z$.

With the above definition, we clearly see that the defining property of the R-time-scale tolerance has been maintained: if $||a||_\infty < \mathcal{E}_t(R)$, then, for all $\varphi \in R$, the stable subspace of $E(\varphi, a)$ is n-dimensional (recall the phase space is $\mathbb{R}^n$) if and only if the same is true for the stable subspace of the impulsive differential equation (2). That is, $E(\varphi, a)$ and (2) have the same stability classification. Therefore the above definition generalizes the associated definitions for periodic equations. Study of the existence of the above generalized time-scale tolerance will not be considered in this article.

4.3.1. A consequence of Theorem 3.2

Theorem 3.2 suggests a method by which a time-scale tolerance can be defined for certain classes of asymptotically stable aperiodic systems, independent of whether or not they possess exponential dichotomies.

Theorem 4.2. *Let R denote a set of impulse extensions for (2) such that, for all $\varphi \in R$, conditions A3 and A4 of Theorem 3.2 hold for all $a \in S^*$, with uniform (σ, w)-regularity in the mean. Assume also that conditions A1, A2 and A5 of Theorem 3.2 are satisfied. For all $t_0 \in \mathbb{R}$, there exists $\delta(R) > 0$ such that, for all $\varphi \in R$, the impulse extension equation $E(\varphi, a)$ is asymptotically stable at t_0 and uniformly attracting on $\mathbb{R}$ whenever $||a||_\infty < \delta(R)$. If $\mathcal{E}_t(R)$ exists, then $\mathcal{E}_t(R) \le \delta(R)$.*

Proof. If one examines the proof of Theorem 3.2, one will notice that the functional representation of φ is never used; only the upper bounds in (20) are needed. Since we have removed the restriction that the bounds are only satisfied in the limit as $a \to 0$, the conclusions of the theorem hold uniformly for all $\varphi \in R$. It follows that there exists $\delta > 0$ such that, if $||a||_\infty < \delta$, $E(\varphi, a)$ is asymptotically stable on $\mathcal{P}(\varphi, a)$, for all $\varphi \in R$. Taking the supremum of all such $\delta > 0$ produces $\delta(R) > 0$.

Suppose $\mathcal{E}_t(R)$ exists. We must have $B_{\mathcal{E}_t(R)}(0) \cap S^* \subseteq \mathcal{E}_s(R)$; otherwise, there would exist $\varphi \in R$ and arbitrarily small $a \in S^*$ with $E(\varphi, a) \sim P$ such that $\text{rank}(I - P) \geq 1$, which would contradict the asymptotic stability of $E(\varphi, a)$ for $||a||_\infty < \delta(R)$. But this implies that, for all $a \in S^*$ with $||a|| < \mathcal{E}_t(R)$, we have $E(\varphi, a) \sim P$ with $\text{rank}(P) = n$, which implies $E(\varphi, a)$ is asymptotically stable. By definition of $\delta(R)$, we obtain $\mathcal{E}_t(R) \leq \delta(R)$. □

As the above theorem demonstrates, under certain conditions, we can define a time-scale tolerance that is, to a certain extent, more optimal than the one provided by Definition 4.7, without resorting to discussions of exponential dichotomies.

4.4. A physical interpretation of uniform exponential regulators

In practice, to compute the time-scale tolerance for a given impulsive differential equation, one must first select a uniform exponential regulator, $R = (\sigma, w)$. There is not much choice over the sequence σ, since the time-scale tolerance may not exist if we do not have $\sigma \in \{\sigma_A, \sigma_F\}$. However, there is much freedom in the choice of w. Recall that the uniform exponential regulator is characterized by inequality (29), which we can write more suggestively as

$$\left|\left|\varphi_k(t,a) - \overline{\varphi_k(t,a)}\right|\right| \leq \frac{w_k(t,a)}{e^{\sigma_k a_k} - 1} \equiv \delta(t, a, w_k),$$

where $\overline{\varphi_k(t,a)}$ is the mean of $\varphi_k(t,a)$ on $\mathcal{S}_k(a)$. As such, the quantity on the right of the inequality represents a functional upper bound for the deviation of φ_k from the mean, on the interval in which the vector field (2) calls it.

For homogeneous systems, we have a fairly simple characterization. If, on the interval $\mathcal{S}_k(a)$, the system evolves according to the differential equation

$$x' = A(t)x + \varphi_k(t,a)x(\tau_k),$$

then the solution satisfies

$$x(t) = \overline{x}(t; x_k) + \text{err}(t, R)x_k,$$

where $\overline{x}(t; x_k)$ is the solution of the IVP

$$x' = A(t)x + \frac{1}{a_k}B_k x(\tau_k), \qquad x(\tau_k) = x_k$$

and $\text{err}(t, R)$ satisfies the inequality

$$||\text{err}(t, R)|| \leq \int_{\tau_k}^{t} \frac{||X^{-1}(s)||w_k(s,a)}{e^{\sigma_k a_k} - 1} ds,$$

where $X' = A(t)X$ and $X(\tau_k) = I$. Interpreting $\overline{x}(t; x_k)$ as the solution of the "impulsively averaged" impulse extension equation, the difference between the true solution, $x(t; x_k)$, and the solution of the averaged equation, $\overline{x}(t; x_k)$, is at most $||\text{err}(t, R)||x_k$ in norm. When $\sigma = \{\sigma_k\}$ is chosen properly (see Theorem 3.1 and associated corollaries), the error tends to zero as $a \to 0$.

From the point of view of applications, this suggests that if φ_k represents some sort of external forcing to the system being modeled, and the forcing acts as a constant under optimal conditions on the duration of the forcing, then one would expect to have $\delta(t, a, w_k) \approx 0$ for $t \in \mathcal{S}_k(a)$ whenever $a_k \geq r_k$ and $[r_k, \Delta\tau_k]$ is the optimal operational range of the forcing function.

If the forcing function is subject to increased error in operation if the duration of the control is less than the minimum of its optimal operational range, one should further expect to have $a_k \mapsto \delta(t, a_k, w_k)$ be strictly decreasing.

If the error associated to the forcing function is ultimately bounded, one would propose

$$\limsup_{a_k \to 0^+} ||\delta(t, a_k, w_k)||_{\mathcal{S}_k(a)}$$

to be finite. On the other hand, if the error of the forcing function is unbounded or, for physical reasons, some range $a_k \in [\tau_k, \tau_k + q_k]$ of durations of impulse effect is not physically attainable (e.g. the forcing function represents the effect of a physical component on the system and is bound by physical constraints), then it would be expected that the above limit superior be infinity.

As such, for different applications, a different choice of w might be more appropriate. One family of functions for which the above limit superior is infinity is given by

$$w_k(a) = C_k(t, a_k) \cdot a_k^{1/\gamma},$$

where $C_k(t, a_k)$ is a continuous and positive on $[\tau_k, \tau_{k+1}] \times [0, \Delta\tau_k]$ and $\gamma > 1$. Choosing C_k and γ carefully, one can ensure the desired monotonicity properties of δ.

5. Discussion

In Section 3, families of (σ, w)-regulated impulse extensions are introduced. It was shown (Theorem 3.1) that the solutions of the impulse extension equation for (1) induced by (φ, a) converge pointwise to the associated solution of the impulsive differential equation as $||a||_\infty \to 0$, provided φ is (σ, w)-regulated. Uniform convergence is also shown to be possible on particular bounded sets. In all cases, the sequence σ must be chosen carefully, but, under certain conditions (Corollary 3.1.2), it can be chosen to be a constant.

Following this, we specialized to periodic equations. Corollary 3.1.3 demonstrated that Floquet multipliers converge to those of the associated impulsive system as the step sequence a becomes small, provided the impulse extension equation is generated by a (σ, w)-regulated family of impulse extensions. Finally, we provided a constructive result for general, aperiodic systems (Theorem 3.2), where the proof was based on Gronwall's inequality and estimations of infinite products.

Section 4 defined the time-scale tolerance, first for asymptotically stable periodic systems (Section 4.1), where an algorithm was provided to compute a lower bound (Algorithm 4.1). This algorithm was discussed in Section 4.1.2, where two methods were suggested to solve a particular optimization problem that is needed to implement the algorithm.

Next, the time-scale tolerance was defined for unstable periodic impulse systems (Section 4.2) for which the monodromy matrix, M_0, satisfied $\rho M_0 > 1$. This problem is more difficult to solve than for asymptotically stable systems, although, assuming one can efficiently minimize the spectral radius map over a compact convex set, Algorithm 4.1 could be adapted to the unstable case.

Finally, we defined the time-scale tolerance for general homogeneous linear systems (Section 4.3) by means of exponential dichotomies. The resulting time-scale tolerance exhibits the same "stability threshold" properties as the analogous specific definitions for periodic systems. Using Theorem 3.2, we proved that, under certain circumstances, one can define a stability threshold for asymptotically stable impulsive systems independently of exponential dichotomies (Theorem 4.2), and the threshold is, in a particular sense, "better" than the time-scale tolerance defined by exponential dichotomies.

All time-scale tolerances are defined with respect to a uniform exponential regulator (Definition 4.1). In Section 4.4, we discussed how uniform exponential regulators should be selected in applications, and their physical interpretation.

Acknowledgments

This research is based on part of the first author's M.Sc. thesis [5] at the University of Ottawa. The second author is supported by an NSERC Discovery Grant. For citation purposes, please note that the question mark in "Smith?" is part of the author's name.

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