ANALYSIS OF PIECEWISE-CONTINUOUS EXTENSIONS OF PERIODIC LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH FIXED, STRICTLY INHOMOGENEOUS IMPULSES

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Abstract. We introduce the notion of impulse extension equations for linear fixed-time impulsive differential equations (IDEs) with strictly inhomogeneous impulses. These differential equations can be thought of as representing the underlying processes for which such linear fixed-time IDEs are a limiting case. We will establish basic existence and uniqueness results, and then develop sufficient conditions for existence and uniqueness of periodic solutions, and study an example of an IDE whose every solution is periodic, even though almost all impulse extension equations which resembles it admit no periodic or bounded solutions.

Keywords. Impulsive differential equation, impulse extension, mathematical model, periodic solution.

AMS (MOS) subject classification: 34A36, 34A37.

1   Introduction

Impulsive differential equations have a host of applications to both biological and physical problems [4, 7, 9, 10, 11, 14, 16]. Classic monographs on the subject [8, 12] preface the exposition of the theory by writing that it is often natural to assume that sufficiently short perturbations in the system occur instantaneously, since their length is negligible in comparison with the duration of the process. A key part of the study of these equations is the existence, uniqueness and stability of their periodic solutions.

   And, indeed, it certainly is convenient to make these assumptions. In particular, for the case of two-dimensional systems, there are very powerful tools available at the hands of the applied mathematician to analyze impulsive differential equations in the plane (see Bainov and Simeonov [2]). Still, it is natural to ask the question: "Is it always safe to assume that sufficiently short processes occur instantaneously?" For example, in a paper by Smith? and Schwartz, the authors looked at an orbitally asymptotically stable periodic orbit of an impulsive differential equation that modelled a theoretical

HIV vaccination regimen involving cytoxic T-lymphocytes (CTLs), and then used numerical simulations to compare it to the case where the activation of the CTLs was nonimpulsive and delayed by up to fourteen days (on a scale of 120-day gap between CTL doses) [13]. Numerical simulations suggested that, in this situation, the impulsive periodic orbit closely resembled the nonimpulsive periodic orbit at almost all times.

Many studies in ecology investigate how pulses of increased resource availability can influence ecological process at different scales; for example, individual, population and community scales. Yang et al. [19] attempt to unify this class of events called resource pulses according to their common underlying processes, which are events characterized by increased resource availability that combine low frequency, short duration and large magnitude. They observe that clear distinctions between resource pulses and background resource variability are generally not possible, and comment that system-specific definitions of resource pulses (e.g., using quantile cutoffs to identify rainfall pulses) can belie the continuous nature of resource variability [19]. They suggest instead that it may be more useful to study how variation in magnitude, duration and frequency of resource inputs will influence the ecological output. In other words, how does the qualitative description of the input resource affect the output? Holt [6] describes the task as one of understanding how the intrinsic structure of the system governs the time course and magnitude of the system's reponses to the pulse.

This idea of resource pulse in ecology can be abstracted mathematically and viewed through the lens of continuous perturbations of ordinary differential equations, with the impulsive case seen as one specific definition of a pulse with a notably discrete structure. Although the degree of coarseness of an approximation of a nonimpulsive differential equation by an impulsive one is important, we choose first to tackle this question of "approximation" from a broader angle. In particular, if we view an impulsive differential equation as a limiting case of a physical process (eg, as the perturbation time approaches zero), can we be confident that the existence of an impulsive periodic solution guarantees that a periodic solution exists in the original physical process, for sufficiently short perturbation times?

We find that this is not the case in general, at the very least for linear systems with fixed, strictly inhomogeneous impulses. In this manuscript, we introduce the impulse extension equation for linear periodic impulsive differential equations with inhomogeneous fixed-time impulses. Following this, we briefly discuss existence and uniqueness of solutions, and then develop necessary and sufficient conditions for existence and uniqueness of periodic solutions. Using these tools, we show that some physical processes with arbitrarily short impulses may admit no periodic solutions, while the impulsive differential equation that they lead to may have one, or possibly infinitely many.

There are a number of sources that phrase results from continuous dynamical systems in terms of impulsive systems: for example, Bonotto and

Federsen recently proved a Poincaré–Bendixson analogue for impulsive semi-dynamical systems [3]. However, there are to our knowledge no publications that bridge continuous systems with short perturbations and their resulting impulsive systems. Thus, in general, it does not seem obvious that the two systems yield consistent results in terms of periodic solutions. It is our hope that this first work on impulse extension equations brings this question to light.

The paper proceeds in the following manner. Section 2 introduces impulse extensions for linear periodic impulsive differential equations, with the definition of impulse extensions and the resulting extension equations among others in Section 2.1. Existence and uniqueness of solutions is covered in Section 2.2. Section 3 covers periodic solutions of impulse extension equations, with Section 3.1 presenting the canonical form for the solution of an initial value problem, and Section 3.2 developing necessary and sufficient conditions for existence of periodic solutions in a variety of cases. Theorem 3.5 establishes the connection between periodic solutions of linear periodic impulsive differential equations and induced impulse extension equations in the non-critical case, whereas Section 3.2.1 provides a counterexample to this result in the critical case. We conclude with a discussion.

2   Impulse Extensions for Linear Periodic Impulsive Differential Equations

We are first interested in linear impulsive differential equations (IDEs) with fixed impulses:

$$
\begin{aligned}
\frac{dx}{dt} &= A(t)x + f(t) & t \neq \tau_k \\
\Delta x &= B_k x + h_k & t = \tau_k
\end{aligned}
\tag{1}
$$

where $x \in \mathbb{R}^n$, $A \in PC(\mathbb{R}, \mathbb{R}^{n \times n})$, $f \in PC(\mathbb{R}, \mathbb{R}^n)$, $B_k \in \mathbb{R}^{n \times n}$ and $h_k \in \mathbb{R}^n$. Here, $\mathbb{R}^{n \times n}$ denotes the set of $n \times n$ real-valued matrices and $PC(Y, Z)$ is the set of piecewise-continuous functions mapping $Y \to Z$ with discontinuities at points τ_k that are continuous from the left. Additionally, we assume the following periodicity and nondegeneracy conditions hold:

- there exists some $T > 0$ such that $A(t + T) = A(t)$ for all $t \in \mathbb{R}$

- $f(t + T) = f(t)$ for all $t \in \mathbb{R}$

- there exists $q \in \mathbb{Z}$ such that $\tau_{k+q} = \tau_k + T$ for all $k \in \mathbb{Z}$

- the *sequence of impulses* τ_k is strictly increasing and $\lim_{k \to \infty} \tau_k = \infty$

- $h_{k+q} = h_k$ for all $k \in \mathbb{Z}$

- $B_{k+q} = B_k$ for all $k \in \mathbb{Z}$.

Periodic solutions of equations of this type have been studied at length in both the periodic [1, 2, 12] as well as the almost-periodic [15, 17, 18] cases. We are interested in the periodic case. The integer q will be referred to as the *cycle number*, and will always assumed to be minimal[1]. As can be verified by the standard literature, initial value problems of this type admit unique, maximal solutions [2].

At times, it may be convenient to refer to a particular T-periodic IDE (1). Since the periodic functions $f(t)$ and $A(t)$, matrices B_k and h_k, sequence of impulses τ_k, and cycle number q uniquely determine such an IDE, we may refer to an T-periodic linear IDE by an ordered 6-tuple $\left(f, A, B_k, h_k, \tau_k, q\right)$.

2.1   Impulse Extensions

Our goal here is to construct an ordinary differential equation from the impulsive differential equation (1) that carries with it the "structure" of the impulse condition. The difference between the original IDE and this new ODE is that the impulse should last a finite, nonzero amount of time. The conditions that we should have are:

$\langle 1 \rangle$ the amount of time this new, "stretched" or "extended" impulse lasts should be short enough that new impulses do not occur before the previous one has finished

$\langle 2 \rangle$ the impulse "extension" should have the same effect on the system as the original impulse in the absence of system evolution

$\langle 3 \rangle$ the impulse extension should have a periodicity condition equivalent to the original impulse

$\langle 4 \rangle$ the resulting ODE should be linear.

Translating these conditions into mathematical language, we have the following definitions.

Definition 2.1. *If $n \in \mathbb{N}^+$, then a sequence of positive real numbers $\{a_k\}$ with $k \in \mathbb{Z}$, is a step sequence of order n over the sequence of impulses $\{\tau_k\}$ if $\tau_k + a_k < \tau_{k+1}$ and $a_{k+nq} = a_k$ for all $k \in \mathbb{Z}$, where q is the cycle number. In this case, we denote by $\mathcal{S}_k := [\tau_k, \tau_k + a_k)$ the k-th step partition and $\mathcal{S} := \bigcup_{k \in \mathbb{Z}} \mathcal{S}_k$ the step space. If $n = 1$, we refer to the sequence $\{a_k\}$ simply as a step sequence.*

Definition 2.2. *A sequence of functions $\{\varphi_k\} = \{(\varphi_k^B, \varphi_k^h)\}_{k \in \mathbb{Z}}$,*

$$\varphi_k^B : \mathcal{S}_k \to \mathbb{R}^{n \times n},$$
$$\varphi_k^h : \mathcal{S}_k \to \mathbb{R}^n,$$

---

[1]For example, if the system is linear and homogeneous with constant impulses $B_k \equiv B$ and $h_k \equiv c$ at fixed, evenly spaced times so that $\tau_{k+1} = \tau_k + T$ for all k, then $q = 1$.

is an impulse extension *for* (1) *over step sequence* $\{a_k\}$ *of order n if, for every* $k \in \mathbb{Z}$, *the functions* φ_k^J *for* $J \in \{B, h\}$ *are continuous at* τ_k *from the right, continuous on* $\mathcal{S}_k$ *except at a finite number of points, bounded and satisfy the following conditions:*

- $\varphi_k^J(t) = \varphi_{k+nq}^J(t + nT)$ *for* $J \in \{B, h\}$ *and all* $k \in \mathbb{R}$, $t \in I_k$, *where* q *is the cycle number*

- *for* $J \in \{B, h\}$, *we have the integral consistency condition*

$$\int_{\mathcal{S}_k} \varphi_k^J(t)dt = J_k.$$

Definition 2.3. *Given a T-periodic impulsive differential equation of the form* (1), *step sequence* $\{a_k\}$ *of order n over* $\{\tau_k\}$, *and an impulse extension* $\{\varphi_k\}$ *over* $\{a_k\}$, *the* impulse extension equation of order n *(n-IEE) induced by* (a_k, φ_k) *is the nT-periodic piecewise-continuous ordinary differential equation*

$$
\begin{aligned}
\frac{dx}{dt} &= A(t)x + f(t) & t \notin \mathcal{S} \\
\frac{dx}{dt} &= A(t)x + f(t) + \varphi_k^B(t)x + \varphi_k^h(t) & t \in \mathcal{S}_k.
\end{aligned}
\tag{2}
$$

If $n = 1$, we refer to such an equation as a *standard impulse extension equation* (st-IEE). In general, we refer to equations of these types simply as *impulse extension equations* (IEEs). A point of the form $\tau_k \in \mathbb{R}$ may occasionally be referred to as an *impulse starting point*, and a point of the form $\tau_k + a_k \in \mathbb{R}$ as an *impulse endpoint*. An *initial value problem* (IVP) is an IEE together with an *initial condition* $x(t_0) = x_0$.

If $\Psi = (f, A, B_k, h_k, \tau_k, q)$ is a linear T-periodic IDE, then the *nth extension class*, $EC^n(\Psi)$, is the set of all tuples (a_k, φ_k) of step sequences $\{a_k\}$ of order n over $\{\tau_k\}$ and impulse extensions $\{\varphi_k\}$ for Ψ over $\{a_k\}$. An element $u \in EC^n(\Psi)$ *induces* an impulse extension equation of order n by equation (2).

At times, we may refer only to an impulse extension, and not explicitly state a choice of step sequence. For example, we may refer to "the st-IEE with impulse extension $\{\varphi_k\}$". In such a situation, we assume that a step sequence has been fixed beforehand, along with the functions $A(t)$ and $f(t)$ and the impulse times $\{\tau_k\}$.

Remark 1: In the case of st-IEEs ($n=1$), Definition 2.2 ensures that condition $\langle 2 \rangle$ is satisfied, and Definitions 2.1 and 2.2 together imply $\langle 3 \rangle$. Moreover, $\langle 4 \rangle$ is satisfied by linearity of the ODE.

The reason to distinguish between the case of st-IEEs and n-IEEs for $n > 1$ is as follows. If $n > 1$, then the resulting n-IEE violates condition $\langle 3 \rangle$.

However, n-IEEs will be useful when comparing nT-periodic solutions of IDEs to nT-periodic solutions of continuous systems. In particular, we can do this because the analysis of nT-periodic solutions of T-periodic IDEs generally does not utilize the fact that T is the minimal period. In this sense, we can loosely consider the T-periodic IDE as a $W = nT$-periodic IDE, and consider W-periodic solutions. In this case, following Remark 1, an n-IEE would be a more general, but equally suitable, ODE type to study, since it would pass all four conditions as outlined at the beginning of Section 2.1.

An interesting consequence of Definition 2.1, which is in some sense independent of periodicity properties, is that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (\tau_{k+1} - \tau_k) > \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} a_k.$$

This implies that the average distance between consecutive impulse times is bounded below by the average of the sequence a_k. This statement is somewhat self evident if each sequence involved is periodic (since each limit can be expressed as a finite sum), but the aforementioned implication remains valid as long as a_k is positive and $\tau_{k+1} < \tau_k + a_k$.

2.1.1　A Note On Initial Conditions

For simplicity, we will always assume that $t_0 \in \mathbb{R}\backslash\text{int}(\mathcal{S})$ when dealing with initial value problems. In particular, we do not have t_0 in an interval of the form $(\tau_k, \tau_k + a_k)$. Note that, in contrast to impulsive differential equations, it is not necessary to specify an initial condition with respect to one-sided continuity (e.g. $x(t_0^+) = x_0$), since solutions to n-IEEs are continuous everywhere (see Section 2.2). However, in this paper, unless otherwise stated, whenever dealing with an impulsive differential equation, we will always take initial conditions of the form $x(t_0^+) = x_0$, as this is the convention of the standard literature [2].

2.2　Existence and Uniqueness of Solutions

Since (2) does not in general have a continuous right-hand side, we will be interested in solutions of IEEs in the extended sense.

Definition 2.4. *Let (2) be written (for convenience) as $\frac{dx}{dt} = F(t, x)$. The function $\Phi : (a, b) \to \mathbb{R}^n$ is a* solution *to (2) if:*

- *$(t, \Phi(t)) \in (a, b) \times \mathbb{R}^n$ for $t \in (a, b)$,*

- *$\Phi(t)$ is differentiable at every $t \in (a, b)$ except possibly at a finite number of points (which may include τ_k and $\tau_k + a_k$ for $k \in \mathbb{Z}$). Furthermore, wherever $\Phi(t)$ is differentiable, $\frac{d\Phi}{dt}(t) = F(t, \Phi(t))$,*

- *$\Phi(t)$ is everywhere continuous.*

$\Phi(t)$ *is said to be a* solution to the initial value problem $x(t_0) = x_0$ *if* $\Phi(t)$ *is a solution to* (2) *and, additionally,* $\Phi(t_0) = x_0$.

As we would expect, the problem is well-posed; a unique solution exists for every initial value problem in some neighbourhood of the initial condition. This is a consequence of the Carathéodory Existence Theorem, which we state below. A proof is available in [5].

Theorem 2.5. (Carathéodory) *Let $D = (a, b) \times U$ be an open set in $\mathbb{R}^{n+1}$ and $F : D \to R^n$ be a function that satisfies the following three conditions:*

- $F(t, x)$ *is measurable in t for each fixed x*

- $F(t, x)$ *is continuous in x for each fixed t*

- *there exists a Lebesgue-integrable function $m : (a, b) \to \mathbb{R}$ such that $|F(t, x)| \le m(t)$ for all $(t, x) \in D$.*

Then, for any (t_0, x_0) in D, there is a solution $\Phi(t) : A \subset \mathbb{R} \to \mathbb{R}^n$ of the differential equation $\frac{dx}{dt} = F(t, x)$ passing through (t_0, x_0) for which the interval of definition A is maximal. Moreover, if, for every compact set $C \subset D$, there is a Lebesgue-integrable function $m_C(t)$ such that

$$|F(t, x) - F(t, y)| \le m_C(t)|x - y|, \qquad (t, x) \in C, \qquad (t, y) \in C,$$

then the solution $\Phi(t)$ is unique and continuous in A.

The solution $\Phi(t)$ given by Theorem 2.5 is a solution in the extended sense, in that it satisfies the ODE except in a set of zero measure, on which it is not differentiable. Equation (2) clearly satisfies all of the conditions of Theorem 2.5; in this case, the solution guaranteed by the theorem may fail to be differentiable at τ_k, $\tau_k + a_k$ and any other points where $A(t)$, $f(t)$ and $\varphi_k(t)$ are not continuous. Since the right-hand side of (2) has at most a finite number of discontinuities on every compact set, the conditions of Definition 2.4 are satisfied. Furthermore, by linearity of equation (2), solutions can be shown to be exponentially bounded and hence globally defined. As a consequence, we have the following.

Theorem 2.6. *Equation* (2) *has a unique continuous solution for each initial value problem $x(t_0) = x_0$, defined for all $t \in \mathbb{R}$.*

3 Periodic Solutions of IEEs with Strictly Inhomogeneous Impulses

For the remainder of this paper, we will be interested in IDEs (1) and IEEs (2) in which the impulse condition is strictly inhomogeneous. That is, we have $B_k \equiv 0$ and $\varphi_k^B \equiv 0$ for all $k \in \mathbb{Z}$. Thus, from this point forward, impulse extensions will take the form $\{\varphi_k\} = \{(0, \varphi_k^h)\}$, so we will identify $\varphi_k = \varphi_k^h$ for notational convenience.

3.1   Canonical Form of Solutions

Consider the initial value problem

$$
\begin{aligned}
\frac{dx}{dt} &= A(t)x + f(t) & t \notin \mathcal{S} \\
\frac{dx}{dt} &= A(t)x + f(t) + \varphi_k(t) & t \in \mathcal{S}_k \\
x(t_0) &= x_0.
\end{aligned}
\tag{3}
$$

Theorem 3.1. *The unique solution of the initial value problem* (3) *can be expressed in the form*

$$
\omega(t) = X(t)x_0 + X(t) \int_{t_0}^{t} X^{-1}(s)f(s)ds + X(t) \sum_{t_0 \leq \tau_k < t} \int_{\tau_k}^{\hat{m}_k} X^{-1}(s)\varphi_k(s)ds
\tag{4}
$$

where $X(t)$ is a fundamental matrix of solutions of the homogeneous problem $\frac{dx}{dt} = A(t)x$ normalized at $t = t_0$ (i.e., $X(t_0) = I$), and $\hat{m}_k = \min\{\tau_k + a_k, t\}$.

Proof. The continuity of $\omega(t)$ follows from the fact that $X(t)$ is fundamental. Indeed, every column of $X(t)$ is a linearly independent solution to the homogeneous problem whose solutions are continuous by Theorem 2.5, so $X(t)v$ is continuous for every $v \in \mathbb{R}^n$. It remains to show that $\omega(t)$ satisfies the conditions on differentiability.

If $t \notin \mathcal{S}_k$, let $\tau_k + a_k < t < \tau_{k+1}$ for some $k \in \mathbb{Z}$. Then, taking the derivative and noting that $\hat{m}_k = \tau_k + a_k$, we obtain

$$
\begin{aligned}
\frac{d\omega}{dt}(t) &= A(t)X(t)x_0 + \left[A(t)X(t) \int_{t_0}^{t} X^{-1}(s)f(s)ds + f(t) \right] \\
&\quad + A(t)X(t) \sum_{t_0 \leq \tau_k < t} \int_{\tau_k}^{\tau_k + a_k} X^{-1}(s)\varphi_k(s)ds \\
&= A(t)\left[X(t)x_0 + X(t) \int_{t_0}^{t} X^{-1}(s)f(s)ds \right. \\
&\quad \left. + X(t) \sum_{t_0 \leq \tau_k < t} \int_{\tau_k}^{\hat{m}_k} X^{-1}(s)\varphi_k(s)ds \right] + f(t) \\
&= A(t)\omega(t) + f(t).
\end{aligned}
$$

On the other hand, if $t \in \mathcal{S}_k$ for some $k \in \mathbb{Z}$ and $q \leq k - 1$ with $q \in \mathbb{Z}$, we

have $\hat{m}_q = \tau_q + a_q$ and $\hat{m}_k = t$. If φ_k is continuous at t, then

$$\frac{d\omega}{dt}(t) = A(t)X(t)x_0 + \left[A(t)X(t)\int_{t_0}^t X^{-1}(s)f(s)ds + f(t)\right]$$

$$+ \left[A(t)X(t)\sum_{q=0}^{k-1}\int_{\tau_q}^{\tau_q+a_q} X^{-1}(s)\varphi_q(s)ds\right.$$

$$+ \left.A(t)X(t)\int_{\tau_k}^t X^{-1}(s)\varphi_k(s)ds + \varphi_k(t)\right]$$

$$= A(t)\left[X(t)x_0 + X(t)\int_{t_0}^t X^{-1}(s)f(s)ds\right.$$

$$+ \left.X(t)\sum_{t_0 \le \tau_j < t}\int_{\tau_j}^{\hat{m}_j} X^{-1}(s)\varphi_j(s)ds\right] + f(t) + \varphi_k(t)$$

$$= A(t)\omega(t) + f(t) + \varphi_k(t).$$

Hence $\omega(t)$, as given in (4), is a solution to (2). It is clear from the normalization of $X(t)$ that $\omega(t_0) = x_0$, so indeed $\omega(t)$ is a solution to the initial value problem. By Theorem 2.6, this is the only solution. $\square$

Remark 2: The solution to the initial value problem for the impulsive differential equation (1) when $B_k \equiv 0$ can be written in a very similar form, as should be expected. Solutions of the impulsive equation can be written in the following way [2] (p.39):

$$\sigma(t) = X(t)x_0 + X(t)\int_{t_0}^t X^{-1}(s)f(s)ds + X(t)\sum_{t_0 \le \tau_k < t} X^{-1}(\tau_k^+)h_k,$$

where $X(t)$ is the same fundamental matrix appearing in (4). Indeed, when $B_k = 0$ for all k, the fundamental matrix of (1) is the same as that of the homogeneous equation $\dot{x} = A(t)x$.

3.1.1 Impulsive Differential Equations as a Limiting Case of Impulse Extension Equations

Before moving on to periodic solutions, we should show that, in the limiting case when $\varphi_k(t) \equiv \delta_{h_k}(t) := \delta(t - \tau_k) \cdot h_k$, where $\delta(t)$ is the Dirac delta function, the solution of the IEE (3) is identical to the solution of the IDE (1) with strictly inhomogeneous impulses; i.e., $B_k \equiv 0$. By properties of the Dirac delta function, computing the solution $\omega(t)$ with an arbitrary step

sequence a_k and initial condition $x(\tau_0) = x_0$, we have, by Remark 2,

$$\omega(t) = X(t)x_0 + X(t)\int_{t_0}^t X^{-1}(s)f(s)ds + X(t)\sum_{t_0 \le \tau_k < t}\int_{\tau_k}^{\hat{m}_k} X^{-1}(s)\delta_{h_k}(s)ds$$

$$= X(t)x_0 + X(t)\int_{t_0}^t X^{-1}(s)f(s)ds + X(t)\sum_{t_0 \le \tau_k < t} X^{-1}(\tau_k)h_k$$

$$= X(t)x_0 + X(t)\int_{t_0}^t X^{-1}(s)f(s)ds + X(t)\sum_{t_0 \le \tau_k < t} X^{-1}(\tau_k^+)h_k$$

$$= \sigma(t).$$

Thus our generalization is consistent; in the limiting case of a sequence of Dirac impulses, the solution of the IEE is the same as that of the original IDE.

3.2 Periodic Solutions

For the rest of this section, we will assume without loss of generality that the initial condition is $x(0) = x_0$. That is, $t_0 = 0$. Additionally, we assume that $t_0 \in \mathbb{R}\backslash\text{int}(\mathcal{S})$ by the discussion of 2.1.1. Let a T-periodic linear IDE be fixed, and recall the canonical form of the solution for an IEE of order n (2):

$$\omega(t) = X(t)x_0 + X(t)\int_0^t X^{-1}(s)f(s)ds + X(t)\sum_{0 \le \tau_k < t}\int_{\tau_k}^{\hat{m}_k} X^{-1}(s)\varphi_k(s)ds.$$

Due to the periodic construction of the impulse extensions — that is,

$$\varphi_{k+nq}(t + T') = \varphi_k(t)$$

for the cycle number q where $T' = nT$ — we have the following standard results from periodic linear differential equations.

Lemma 3.2. *The solution $\omega(t)$ is T'-periodic if and only if $\omega(0) = \omega(T')$.*

Corollary 1. *The solution $\omega(t)$ is kT'-periodic for some $k \in \mathbb{Z}^+$ if and only if $\omega(0) = \omega(kT')$.*

Evaluating $\omega(t)$ at 0 and kT', we arrive at a necessary and sufficient condition for the existence of a kT'-periodic solution of (2). Moreover, in the case $k = 1$, the condition guarantees uniqueness as well. We omit the proof for brevity, but it is nearly identical to the case for impulsive differential equations [2] (p.39).

Lemma 3.3. *The impulse extension equation (2) admits a kT'-periodic solution if and only if there exists some $x_0 \in \mathbb{R}^n$ that satisfies the equation*

$$\left(I - X(kT')\right)x_0 = X(kT')\left[\int_0^{kT'} X^{-1}(s)f(s)ds + \sum_{0 \le \tau_r < kT'}\int_{\mathcal{S}_r} X^{-1}(s)\varphi_r(s)ds\right].$$

$$(5)$$

If such an x_0 exists, then the kT'-periodic solution is the solution to the initial value problem $x(0) = x_0$, and is given by equation (4). If equation (5) has a solution for $k = 1$, then this solution is the unique T'-periodic solution.

If $I - X(kT')$ is invertible, then it is clear that the solution x_0 exists; this is the "non-critical case". If $I - X(kT')$ is singular — the "critical case" — then the condition for existence of a periodic solution takes a different form. We summarize this with the following theorem, whose proof is nearly identical to that for IDEs, and the reader may refer to [2] (p.44) to see the mechanics at work.

Theorem 3.4. *Equation* (2) *admits a kT'-periodic solution if and only if* $\det(I - X(kT')) \neq 0$ *or*

$$\int_0^{kT'} Y_j^*(t)f(t)dt + \sum_{0 \leq \tau_r < kT} \int_{\mathcal{S}_r} Y_j^*(t)\varphi_r(t)dt = 0 \tag{6}$$

for $j = 1, \ldots, m$, where $*$ denotes the tranpose, m is the number of linearly independent kT'-periodic solutions of the homogeneous equation and Y_j is the jth column of the fundamental matrix of the adjoint equation,

$$\frac{dy}{dt} = -A^*(t),$$

normalized at $t_0 = 0$.

The following, which is largely a corollary to Theorem 3.4, utilizes the fact that linear T-periodic IDEs in the non-critical case always have periodic solutions.

Theorem 3.5. *Let $X(t)$ be the normalized fundamental matrix of the homogeneous equation $\frac{dx}{dt} = A(t)x$ for the T-periodic IDE Ψ, and suppose $\det(I - X(kT)) \neq 0$ for some positive integer k. Then the T-periodic IDE Ψ admits a kT-periodic solution; furthermore, if $k = 1$, there is a unique T-periodic solution. If $r \in \mathbb{N}^+$ divides k, then every element of $EC^r(\Psi)$ induces an IEE of order r with a kT-periodic solution; if $r = k$, there is a unique kT-periodic solution.*

Proof. Existence of a kT-periodic solution of the IDE Ψ is proven in [2], and uniqueness for $k = 1$ is given there as well.

If r divides k, then $k = rd$ for some integer d. If $u \in EC^r(\Psi)$, then u induces an IEE of order r and period $rT = T'$. But $kT = drT = dT'$, so $\det(I - X(dT')) = \det(I - X(kT)) \neq 0$. By Theorem 3.4, the induced IEE has a $dT' = kT$-periodic solution. If $r = k$, then $rT = kT = T'$, so, by Theorem 3.4, the induced IEE has a unique $T' = kT$-periodic solution. $\quad\square$

3.2.1   Example: Simple Harmonic Oscillator With Impulses

We give an example of an impulsive differential equation whose every solution is 2π-periodic, but for which there exists an impulse extension and step sequence whose corresponding IEE has no periodic solution. Consider the π-periodic impulsive differential equation

$$
\begin{array}{lll}
x' = y, & y' = -x, & t \neq k\pi \\
\Delta x' = 0, & \Delta y = c, & t = k\pi
\end{array}
\tag{7}
$$

where $c \neq 0$ and the impulses occur at moments $\tau_k = k\pi$. We will consider the existence and uniqueness of 2π-periodic solutions for this IDE, as well as a particular IEE derived from it.

Part 1: Impulsive Differential Equation

Writing the homogeneous equation in matrix form

$$
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A(t) \begin{bmatrix} x \\ y \end{bmatrix},
$$

we see that the matrix $A(t)$ is self-adjoint, so the fundamental matrix of both the homogeneous and adjoint equation is

$$
X(t) = Y(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.
$$

We have $I - X(2\pi) = I - I = 0$, so we are in the critical case. The compatibility conditions [2] (p.44) for existence of a 2π-periodic solution in the critical case take the form

$$
Y_1^*(0)h_k + Y_1^*(\pi)h_k = 0
$$
$$
Y_2^*(0)h_k + Y_2^*(\pi)h_k = 0.
$$

However, we do not even need to compute these, since a direct verification of the general form of the solution shows that every solution is 2π-periodic. Indeed, since $X(2\pi) = I$, we have

$$
\begin{aligned}
\omega(T) &= X(2\pi)x_0 + \sum_{0 \leq \tau_k < T} X(\tau_k) \cdot h_k \\
&= I \cdot x_0 + \begin{bmatrix} \cos(0) & \sin(0) \\ -\sin(0) & \cos(0) \end{bmatrix} h_k + \begin{bmatrix} \cos(\pi) & \sin(\pi) \\ -\sin(\pi) & \cos(\pi) \end{bmatrix} h_k \\
&= x_0 + h_k - h_k \\
&= x_0 \\
&= \omega(0).
\end{aligned}
$$

Therefore, every solution of the impulsive differential equation is 2π-periodic (see, for example, Figure 1). Furthermore, since $X(\pi) = -I$, it will also have
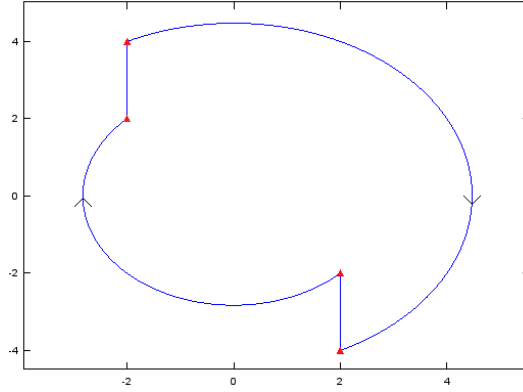
Figure 1: Impulsive 2π-periodic orbit of equation (7) with $c = 2$ and $x(0^+) = x_0 = [2, -2]^T$. Impulse points and endpoints are displayed using triangles, and direction is given by arrowheads.

a unique π-periodic solution by Lemma 3.3.

Part 2: Impulse Extension Equation
The cycle number of the IDE is $q = 1$, since $\tau_{k+1} = \tau_k + \pi$. We will now construct a step sequence and an impulse extension of order 2:

$$a_k = \left\{ \begin{array}{ll} \pi/2 & \text{if } k \text{ is even} \\ \pi/3 & \text{if } k \text{ is odd} \end{array} \right. \qquad \varphi_k(t) = \left[\begin{array}{c} 0 \\ c/a_k \end{array} \right]. \qquad (8)$$

We have $a_{k+2q} = a_{k+2} = a_k$, so $\{a_k\}$ is a step sequence of order 2. It can be verified that $(a_k, \varphi_k) \in EC^2$ for this IDE, so the induced IEE has period 2π. For 2π-periodic solutions, we are in the critical case again, so we must examine the compatibility condition (6). For this system, it is

$$\int_0^{a_0} Y_j^*(t)\varphi_0(t)dt + \int_\pi^{\pi+a_1} Y_j^*(t)\varphi_1(t)dt = 0 \qquad\qquad j = 1, 2,$$

which becomes, after substituting in the step sequence and impulse extension (a_k, φ_k),

$$\int_0^{\pi/2} Y_j^*(t) \left[\begin{array}{c} 0 \\ 2c/\pi \end{array} \right] dt + \int_\pi^{4\pi/3} Y_j^*(t) \left[\begin{array}{c} 0 \\ 3c/\pi \end{array} \right] dt = 0 \qquad\qquad j = 1, 2.$$

Note that $\varphi_k(t)$ is not constant; it depends on the sequence a_k. For j=1, we

have

$$\int_0^{\pi/2} \begin{bmatrix} \cos(t) & -\sin(t) \end{bmatrix} \begin{bmatrix} 0 \\ 2c/\pi \end{bmatrix} dt = -\frac{2c}{\pi}$$

$$\int_\pi^{4\pi/3} \begin{bmatrix} \cos(t) & -\sin(t) \end{bmatrix} \begin{bmatrix} 0 \\ 3c/\pi \end{bmatrix} dt = \frac{3\sqrt{3}c}{2\pi}$$

and so

$$\int_0^{\pi/2} Y_1^*(t) \begin{bmatrix} 0 \\ 2c/\pi \end{bmatrix} dt + \int_\pi^{4\pi/3} Y_1^*(t) \begin{bmatrix} 0 \\ 3c/\pi \end{bmatrix} dt = \frac{c(3\sqrt{3}-2)}{2\pi} \neq 0.$$

Therefore the impulse extension equation has no 2π-periodic solutions.

In conclusion, although every solution of the impulsive differential equation is 2π-periodic, the pair $(a_k, \varphi_k) \in EC^2$ introduced in (8) induces an IEE with no 2π-periodic solution. This is no coincidence; consider now the step sequence and impulse extension

$$\bar{a}_k = \begin{cases} r_0 & \text{if } k \text{ is even} \\ r_1 & \text{if } k \text{ is odd} \end{cases} \qquad \bar{\varphi}_k(t) = \begin{bmatrix} 0 \\ c/a_k \end{bmatrix} \qquad (9)$$

where r_0 and r_1 are fixed (not necessarily distinct) real numbers between 0 and π (so as to satisfy the definition of a step sequence). If $r_0 = r_1$, then $(\bar{a}_k, \bar{\varphi}_k) \in EC^1$; $(\bar{a}_k, \bar{\varphi}_k) \in EC^2$. The compatibility condition for the existence of a 2π-periodic solution of the induced IEE is

$$\int_0^{r_0} Y_j^*(t) \begin{bmatrix} 0 \\ c/r_0 \end{bmatrix} dt + \int_\pi^{\pi+r_1} Y_j^*(t) \begin{bmatrix} 0 \\ c/r_1 \end{bmatrix} dt = 0 \qquad j = 1, 2.$$

After some manipulations, the condition becomes

$$\frac{c}{r_0} \int_0^{r_0} \sin(t) dt + \frac{c}{r_1} \int_\pi^{\pi+r_1} \sin(t) dt = 0$$

$$\frac{c}{r_0} \int_0^{r_0} \cos(t) dt + \frac{c}{r_1} \int_\pi^{\pi+r_1} \cos(t) dt = 0.$$

Calculating these integrals, we see that the IEE has a 2π-periodic solution if and only if

$$\frac{\cos(r_0)}{r_0} + \frac{\cos(\pi + r_1)}{r_1} = 0$$
$$\frac{\sin(r_0)}{r_0} + \frac{\sin(\pi + r_1)}{r_1} = 0. \qquad (10)$$

Recall that sine and cosine are each antiperiodic with antiperiod π. After some simple manipulations, we arrive at a necessary and sufficient condition for the existence of a 2π-periodic solution when $r_0 \neq \pi/2$;

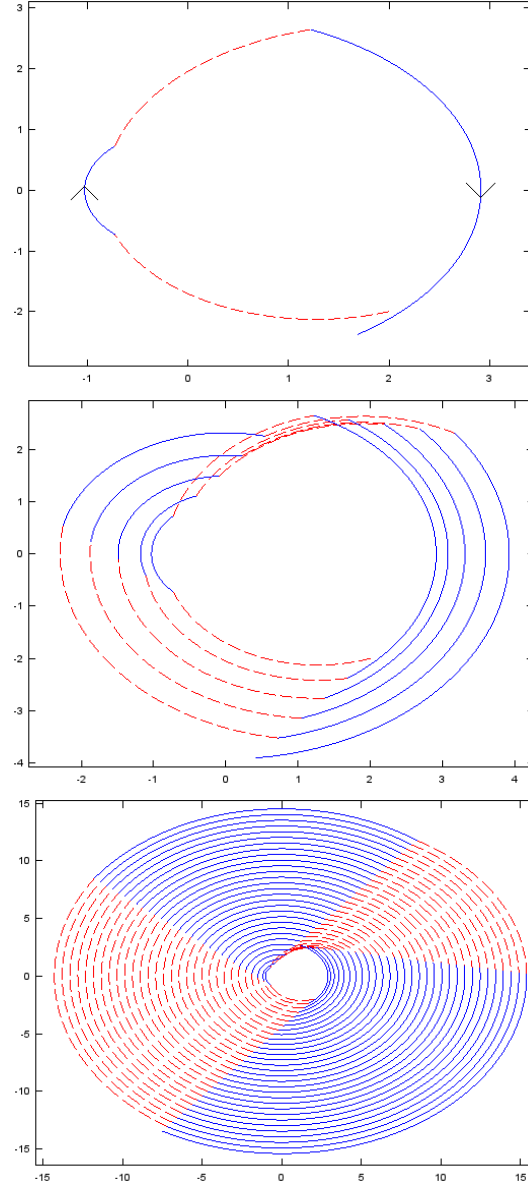$$\frac{r_1}{r_0} = \frac{\cos(r_1)}{\cos(r_0)} = \frac{\sin(r_1)}{\sin(r_0)}.$$

Figure 2: Forward orbits from $x_0 = [2, -2]^T$ with $c = 2$ of the IEE under step sequence and extension function (a_k, φ_k) of equation (8). Orbits during the impulse extension are represented by dashed lines. From top to bottom, orbits are evaluated at $t = 2\pi$, $t = 10\pi$, $t = 60\pi$. Arrowheads in the top figure indicate the direction of the orbit; the direction is the same in the other two figures.
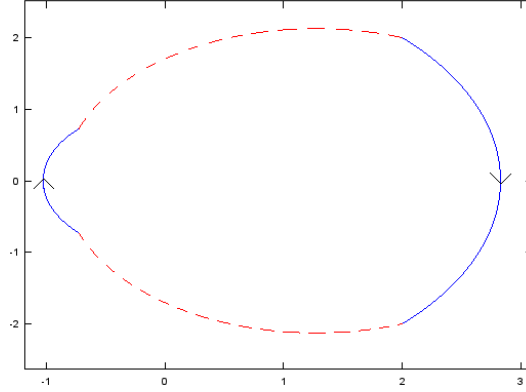
Figure 3: Periodic orbit of the IEE starting at $x_0 = [2, -2]^T$ with step sequence and impulse extension $(\bar{a}_k, \bar{\varphi}_k)$ of (9) with $c = 2$ and $r_0 = r_1 = \frac{\pi}{2}$. Orbits during the impulse extension are represented by dashed lines. The direction is indicated by the arrowheads.

Multiplying through by the sines and cosines, this reduces to

$$\sin(r_0)\cos(r_1) - \cos(r_0)\sin(r_1) = 0.$$

But this is just the angle difference formula for the sine function. Hence, when $r_0 \neq \pi/2$, a 2π-periodic solution exists if and only if $\sin(r_0 - r_1) = 0$. Since, by hypothesis, we required $0 < r_j < \pi$ for $j = 1, 2$, and since sine has only one zero in $(-\pi, \pi)$, this is equivalent to $r_0 = r_1$. See Figure 2 for an example of an unbounded solution and Figure 3 for a 2π-periodic solution.

When $r_0 = \pi/2$, the first line of equation (10) implies that $\cos(r_1) = 0$. Again, since we have restricted r_1 to the interval $(0, \pi)$, we must have $r_1 = \pi/2$ for this equation to be satisfied. Hence, the conclusion is the same as when $r_0 \neq \pi/2$; a 2π-periodic solution exists if and only if $r_0 = r_1$. Furthermore, a direct verification of the solution confirms that if $r_0 = r_1$, then every solution of the IEE is 2π-periodic, and if $r_0 \neq r_1$, then every solution is unbounded. Additionally, if $r_0 = r_1$, then we can consider $(\bar{a}_k, \bar{\varphi}_k) \in EC^1$; by Theorem 3.5, the induced IEE will have a unique π-periodic solution as well.

In fact, the result is stronger still. When $r_0 \neq r_1$, the IEE admits no periodic solution under the step sequence and impulse extension $(\bar{a}_k, \bar{\varphi}_k)$ at all. To see this, first note that, by the construction of the impulses (occuring at each unit of time $k\pi$), we only need to consider periods of length $k\pi$ for an integer k. Clearly, there are no such $k\pi$-periodic solutions when k is odd. In the case where k is even, $X(k\pi) = I$, so we are still in the critical case. It is easy to see that the compatibility condition is a simple modification of

(10):

$$\sum_{n=0}^{(k/2)-1} \frac{\cos(2n\pi + r_0)}{r_0} + \frac{\cos((2n+1)\pi + r_1)}{r_1} = 0$$

$$\sum_{n=0}^{(k/2)-1} \frac{\sin(2n\pi + r_0)}{r_0} + \frac{\sin((2n+1)\pi + r_1)}{r_1} = 0.$$

By periodicity, after computing the sum, this is equivalent to

$$\frac{k}{2}\left(\frac{\cos(r_0)}{r_0} + \frac{\cos(\pi + r_1)}{r_1}\right) = 0$$

$$\frac{k}{2}\left(\frac{\sin(r_0)}{r_0} + \frac{\sin(\pi + r_1)}{r_1}\right) = 0.$$

Since $k > 0$, we can divide by $k/2$. Therefore, the above holds if and only if $r_0 = r_1$, just as before. We arrive at the following conclusions:

1. The impulsive differential equation (7) has a unique π-periodic solution, and all of its other solutions are 2π-periodic

2. The IEE induced by $(\bar{a}_k, \bar{\varphi}_k)$ given in equation (8) has a periodic solution if and only if $r_0 = r_1$. If this holds, then there is a unique π-periodic solution, and every other solution is 2π-periodic. If this does not hold, then every solution is unbounded.

4   Discussion

In Section 2, we introduced impulse extensions for general linear fixed-time impulsive differential equations and showed that these differential equations have unique, globally defined solutions. In Section 3, we limited the scope of the discussion to IEEs with strictly inhomogeneous impulses. We expect that results analogous to, for example, Theorem 3.5, hold in the general case as introduced in Section 2.

Theorem 3.5 provides a criterion that guarantees that an impulsive differential equation and all impulse extension equations conjugate to it will have (under certain circumstances, unique) periodic solutions.

We have seen in Example 3.2.1 that it is possible for an impulsive differential equation to admit infinitely many periodic solutions, even though there exists a family of piecewise-continuous impulse extension equations that are conjugate to it for which none of its members admit periodic solutions. In particular, the result in this example holds for arbitrarily short lengths of impulsive extension. With respect to periodic solutions, it is therefore possible for a continuous system with arbitrarily short periods of continuous impulse extension to never resemble the impulsive system to which it is conjugate.

This example fell into the critical case where the hypothesis of Theorem 3.5 failed to hold. We may ask if this same type of behaviour persists in general when this hypothesis (i.e. $\det(X(kT) - I) \neq 0$) is not satisfied.

Acknowledgements

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