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Classification of planar Filippov systems with a delayed threshold: Applications to infectious disease transmission dynamics ☆

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ABSTRACT

In classical Filippov systems, the switching between vector fields is typically assumed to occur instantaneously upon crossing the switching manifold. In practice, however, there is inevitably a time delay between observing the system's state and implementing control actions, such as delays in reporting infection data and enacting interventions in epidemiological models. Until now, a general classification of planar Filippov systems with a single switch has not been developed. By employing a Poincaré map and asymptotic matching techniques, we investigate the properties of periodic solutions induced by time delays, including their uniqueness, stability, amplitude and location in the state space. We then apply this framework to an epidemiological model and analyse its global dynamics. Our results show that, under certain sufficient conditions, the epidemic system stabilizes either at an equilibrium of subsystems or at a periodic orbit induced by the delayed threshold policy, depending on the chosen threshold levels. If the threshold level is set too high or too low, the delay has little effect on the long-term dynamics; in contrast, intermediate thresholds may lead to oscillations in the disease cycle. These findings highlight that timely reporting of infection data, prompt implementation of control measures and careful selection of threshold levels are critical for mitigating the spread of infectious diseases.

1. Introduction

Piecewise-smooth differential dynamical systems have gained significant attention, owing to their wide scope of applications across diverse fields, including mechanical engineering [1,2], control theory [3] and mathematical biology [4–7]. The dynamics of such systems are characterized by the smooth evolution interrupted by instantaneous events. Simpson [8] identified twenty distinct mechanisms that can locally generate limit cycles in planar piecewise-smooth systems and provided detailed illustrations in subsequent work [9]. Depending on the degree of smoothness across the switching manifold, these systems can be classified into three categories: systems with uniform smoothness of degree two or higher, degree one and degree zero — corresponding to piecewise-smooth continuous systems, Filippov systems and hybrid dynamical systems, respectively [10]. The present study focuses on Filippov systems and their applications in mathematical biology.

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The non-smoothness or discontinuity inherent in such systems often leads to mathematical challenges, including incompatible behaviour of vector fields near switching manifolds and the emergence of diverse bifurcation phenomena [11]. From a theoretical perspective, numerous studies have investigated the rich dynamical behaviours [12–19] and the associated regularized problems [20, 21] in Filippov systems. Here, regularization refers to replacing the original piecewise-smooth system with an appropriately smooth differential system in a neighbourhood of the switching manifold. In planar Filippov systems, Kuznetsov et al. [12] provided a partial classification of codimension-1 local bifurcations, including explicit normal forms and certain global bifurcations. The case of missing boundary-equilibrium bifurcations was further addressed by Hogen et al. [22]. Guardia et al. [14] described generic codimension-2 singularities and their associated dynamics, while Efstathiou et al. [17] analysed the codimension-3 boundary-Hopf-fold (BHF) bifurcation and derived eight corresponding bifurcation diagrams using a local normal form. From an applied standpoint, Filippov systems have been widely used to study the transmission dynamics of infectious diseases [23–25]. Guo et al. [26] introduced a discontinuous treatment strategy into an SIR epidemic model and demonstrated that such a strategy could facilitate disease elimination more effectively than continuous interventions. Wang et al. [27] applied a threshold policy to the problem of medical-resource constraints, showing that multistability of three equilibria could be induced as the threshold varies. Xiao et al. [5] analysed an SIR model with a piecewise control function based on a threshold policy and showed that an appropriate choice of thresholds and control efforts can either prevent disease outbreaks or maintain the infection level within a manageable range. Wang et al. [28] used a threshold policy to understand the amplification of social-distancing in epidemics, showing that discontinuity-induced bifurcations could lead to multistability of three equilibria.

In these models, the switching between vector fields is typically assumed to occur instantaneously upon crossing the switching manifold. In reality, however, there is inevitably a time delay between observing the system state and implementing control measures, such as delays in reporting infection numbers and enacting interventions in infectious disease models [11,29]. Moreover, it is well known that when such time delays are neglected, the trajectory of a Filippov system may exhibit sliding along the switching manifold, characterized by a rapid succession of switches within a short time interval [30]. In other words, as the switching frequency tends to infinity, this rapid switching behaviour approximates a sliding motion. In contrast, during real infectious disease transmission, the frequency of switching control strategies is finite, causing the system state to “chatter” near the switching manifold. Based on the SIR switching model [5], Muqbel et al. [29] proposed that control measures are implemented with a finite time delay after the density of infected individuals exceeds a prescribed threshold. Their results provided valuable insights into disease control by examining the interplay between control intensity, threshold levels and the delay in implementing interventions. Wang et al. [11] further investigated a Filippov SIQR model with delayed relay control and derived sufficient conditions to ensure the uniqueness and global stability of a slowly oscillating periodic solution.

Motivated by these studies, we investigate the properties of periodic solutions in a more general Filippov system that incorporates a time delay on the switching manifold. Our analysis begins with the classical Filippov system (without delay), providing a natural and novel perspective for comparison. This approach highlights both the distinctions and the connections between planar Filippov systems with delayed switching and their classical, instantaneous counterparts. Furthermore, it is of particular interest to determine the location of the resulting periodic solutions in the phase plane. To this end, we consider a general class of planar Filippov systems with a time delay on the switching manifold:

$$x'(t) = \begin{cases} f_1(x(t)) & \text{if } h(x(t-\tau)) < 0, \\ f_2(x(t)) & \text{if } h(x(t-\tau)) > 0, \end{cases} \tag{1.1}$$

where $x = [x_1, x_2]^T$ and $f_1(x) = [f_{11}, f_{12}]^T, f_2(x) = [f_{21}, f_{22}]^T : \mathbb{R}^2 \hookrightarrow \mathbb{R}^2$ are locally Lipschitz continuous. The switching manifold is defined by $h(x) = x_2 - k$, where $k > 0$ denotes the threshold for implementing control strategies. The parameter τ represents the time delay in switching decision. Owing to the presence of τ in (1.1), the initial data space is $\mathbb{R}_+ \times C$, where $C \equiv C([-\tau, 0], \mathbb{R}_+)$ is the Banach space of continuous functions on $[-\tau, 0]$. Moreover, both $f_1(x(t))$ and $f_2(x(t))$ must be well defined throughout $\mathbb{R}^2_+$. Beyond these structural differences, the introduction of a time delay has a significant impact on the system dynamics compared to the classical planar Filippov system ($\tau = 0$), and analysing this impact is the main objective of the present study. The main contributions of this paper can be summarized as follows:

(i) We classify planar Filippov systems with a delay that represents the realistic time delay between observing the system's state and implementing control actions.
(ii) We analyse the distinctions and connections between Filippov systems with time delay and classical (instantaneous) Filippov systems.
(iii) We apply the proposed Filippov framework with a delayed threshold policy to a specific SIR epidemic model and further examine how the delay influences the system's dynamic behaviour.

The rest of this paper is outlined as follows. In Section 2, we define the solution of (1.1) satisfying initial data using the method of steps, and we establish the existence, uniqueness and non-negativity of the solution. Section 3 provides a brief review of fundamental concepts for the classical planar Filippov system (1.1) ($\tau = 0$) and their generalization to the system with time delay ($\tau > 0$). Moreover, periodic solutions are investigated via an appropriate Poincaré map. In Section 4, we construct a specific SIR model incorporating a time delay in switching decision and analyse its global dynamics. Numerical simulations illustrating the main results are presented in Section 5. Finally, concluding remarks and discussions on planar Filippov system (1.1) are provided in the last section.

2. Well-posedness

Due to the discontinuity on the right-hand side of system (1.1), the standard definition of a solution to the differential equation with continuous right-hand side is not directly applicable. To ensure the solution of system (1.1) is mathematically well-defined, we choose the initial state space as $\Omega = \mathbb{R}_+ \times X$, where

$$X = \{x_2(\theta) \in C \mid x_2(\theta) - k = 0 \text{ has finitely many roots on } [-\tau, 0] : t_1 < \cdots < t_n \leqslant 0\}.$$

We denote the solution semiflows corresponding to the lower subsystem $x'(t) = f_1(x(t))$ and the upper subsystem $x'(t) = f_2(x(t))$ by $\Phi_1 = (\Phi_{11}, \Phi_{12})$ and $\Phi_2 = (\Phi_{21}, \Phi_{22})$, respectively. In the subsequent analysis, solutions are defined using the method of steps, and the existence and uniqueness of solutions are established.

Theorem 2.1. *For any initial condition $x_0 = (x_{10}, x_2(\theta)) \in \Omega$, system (1.1) admits a unique solution $x(t; x_0)$ on the interval $[-\tau, \hat{T})$, where $0 < \hat{T} \leqslant +\infty$. Moreover, the solution is dependent continuously on $x_{10}, x_2(0)$, and switching moments $t_1, \ldots, t_n$.*

Proof. We first construct the solution of the system with the initial condition $x_0 \in \Omega$ over the interval $(0, \tau]$ and then extend it to $(m\tau, (m+1)\tau], m \in \mathbb{N}^+$ through iteration. Since $x_2(\theta) \in X$, the interval $(0, \tau]$ can be decomposed into $n+1$ subintervals as

$$J_l = (t_l + \tau, t_{l+1} + \tau] \quad \text{for } l = 0, \ldots, n,$$

where $t_0 = -\tau$ and $t_{n+1} = 0$. Within each subinterval J_l, the forward evolution of system (1.1) follows one of the flows $\Phi_{j_l} (j_l = 1, 2$ and $l = 0, \ldots, n)$. Hence, the solution of system (1.1) for $t \in [0, \tau]$ can be obtained by successively concatenating these flows; i.e.,

$$x(t) = \begin{cases} \Phi_{j_0}(t; x_{10}, x_2(0)) & \text{for } t \in J_0, \\ \Phi_{j_l}(t - t_l - \tau; x(t_l + \tau)) & \text{for } t \in J_l,\ l = 1, \ldots, n. \end{cases} \tag{2.1}$$

The forward evolution $x_t(\theta; x_0)$ for $t \in (0, \tau]$ is defined as

$$x_t(\theta; x_0) = \begin{cases} x(t + \theta) & \text{for } \theta \in (-t, 0], \\ x_0(t + \theta) & \text{for } \theta \in [-\tau, -t], \end{cases} \tag{2.2}$$

where $x_t(\theta; x_0) = x(t + \theta; x_0)$.

When $t > \tau$, the forward evolution $x_t(\theta; x_0)$ is recursively defined by

$$x_t(\theta; \cdot) = x_{t/(m+1)}(\theta; \cdot) \circ \cdots \circ x_{t/(m+1)}(\theta; \cdot), \qquad t \in (m\tau, (m+1)\tau], \tag{2.3}$$

where $\circ$ denotes the concatenation of successive evolutions and $m \in \mathbb{N}^+$. Together with (2.1)–(2.3), we establish the uniqueness of the solution of system (1.1) on $[-\tau, \hat{T})$, the recursive relation (2.1) and the switching moments $t_1, \ldots, t_n$. Hence, the initial-value problem is intrinsically finite-dimensional, even though the initial data on $[-\tau, 0]$ are required to define the solution of system (1.1) on $[0, +\infty)$. □

Remark. The uniqueness of the solution for system (1.1) with the initial condition $x_0 \in \bar{\Omega}$ arises from the introduction of a time delay τ, which effectively separates the original switching line into two distinct lines (see Figs. 1 and 4(b)), similar to the two-thresholds model in [31]. As the time delay tends to zero, these two lines coalesce into a single switching line (see Fig. 4(a)), and it is well known that the solution then loses uniqueness. Furthermore, it should be emphasized that if the initial condition $x_0 = (x_{10}, x_2(\theta))$ satisfies $x_2(\theta) \equiv k$ for $\theta \in [\tau_1, \tau_2]$, where $-\tau \leqslant \tau_1 < \tau_2 \leqslant 0$, then the uniqueness of solutions for system (1.1) stated in Theorem 2.1 no longer holds. It should be noted that, in general, uniqueness in Filippov systems is not expected, due to the nature of the threshold dynamics [32,33].

Note that the local solution of system (1.1) for any $x_0 \in \Omega$ is guaranteed by Theorem 2.1. This local solution can be extended in a manner analogous to the standard extension procedure for ordinary differential equations. The uniqueness of solutions implies that if $x : [-\tau, \hat{T}_1\rangle \hookrightarrow \mathbb{R}^2$ and $\hat{x} : [-\tau, \hat{T}_2\rangle \hookrightarrow \mathbb{R}^2$ are two solutions of system (1.1) (where $[-\tau, \cdot\rangle$ denotes $[-\tau, \cdot)$ or $[-\tau, \cdot]$), then $x(t) = \hat{x}(t)$ whenever both are defined at the same t. If $[-\tau, \hat{T}_1\rangle \subset [-\tau, \hat{T}_2\rangle$, we write $x \subset \hat{x}$ to indicate that $\hat{x}$ is an extension (or continuation) of x. This inclusion relation defines a partial order on the set of all solutions to (1.1). By applying Zorn's Lemma [34], we conclude the existence of a unique maximal (noncontinuable) solution. Such a solution is defined on an interval with an open right endpoint. Otherwise, if it could be extended beyond its current domain, Theorem 2.1 would yield a larger interval of existence, contradicting the assumption of noncontinuability.

The following result shows that if a noncontinuable solution is not defined on the entire interval $[-\tau, +\infty)$, the solution must blow up as $t \to \hat{T}^-$.

Theorem 2.2. *Let $x : [-\tau, \hat{T}) \hookrightarrow \mathbb{R}^2$ be a noncontinuable solution of system (1.1) with initial condition $x_0 \in \Omega$ and $\hat{T} < +\infty$. Then*

$$\lim_{t \to \hat{T}^-} \|x(t)\| = +\infty.$$

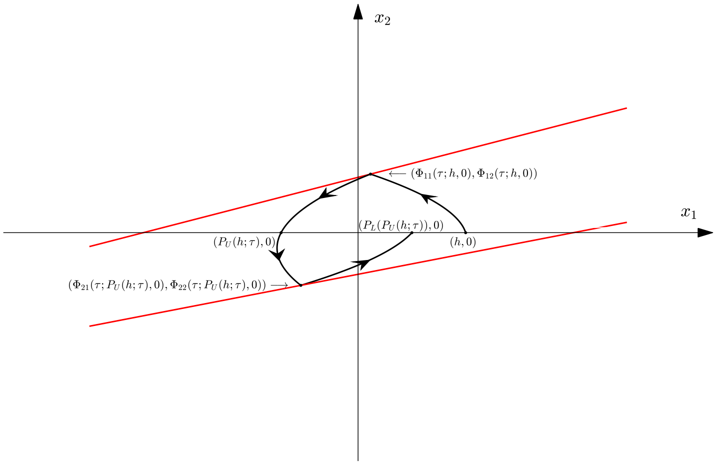


Fig. 1. A schematic illustration of the Poincaré map for system (1.1), where the Poincaré section is taken at $x_2 = 0$. The red lines denote the locations where the vector field of system (1.1) switches, which now depend on the time delay τ and therefore no longer coincide with the original switching line $x_2 = 0$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Proof. Assume that $x(t)$ is a noncontinuable solution and $\hat{T} < +\infty$. Then there exists $\bar{t}_1$ such that the restriction of $x(t)$ to $[\bar{t}_1, \hat{T})$ is also a noncontinuable solution of the following initial-value problem:

$$\begin{aligned} &y'(t) = f_i(y(t)), \\ &y(\bar{t}_1) = x(\bar{t}_1), \end{aligned} \tag{2.4}$$

where $i = 1$ or 2. Note that any continuation of (2.4) would yield a continuation of the solution to (1.1) with $x_0 \in \Omega$, which contradicts the noncontinuability of $x(t)$. Therefore, by the continuation theorem for ordinary differential equations [35], it follows that $\lim_{t\to\hat{T}^-} \|x(t)\| = +\infty$. $\square$

The non-negativity of solutions with initial condition $x_0 \in \Omega$ is discussed next. We use the notation $x \geqslant 0$ to denote $x_i \geqslant 0$ for $i = 1, 2$.

Theorem 2.3. *Assume that $x_0 \in \Omega$ and that for all $i \in \{1, 2\}$ and $t, x \in \mathbb{R}^2_+$,*

$$x_i = 0 \Longrightarrow f_{1i}(x) \geqslant 0, f_{2i}(x) \geqslant 0.$$

Then the solution $x(t)$ of system (1.1) satisfies $x(t) \geqslant 0$ for all $t \geqslant 0$.

Proof. By combining the corresponding result for ordinary differential equations (see Proposition B.7 in [36]) with the definition of the solution given in (2.1)–(2.3) for the solution of system (1.1), it follows directly that $x(t) \geqslant 0$ for all $t \geqslant 0$, and consequently $x_t(\theta; x_0) \in \mathbb{R}_+ \times C$. $\square$

3. The dynamic behaviour of system (1.1)

To better investigate the effect of the introduced time delay τ on system (1.1), we first briefly review the fundamental theory related to the classical planar Filippov system (1.1) with $\tau = 0$.

3.1. *The planar Filippov system (1.1) with $\tau = 0$*

When $\tau = 0$, the Filippov system (1.1) can be rewritten as

$$x'(t) = \begin{cases} f_1(x(t)) & \text{if } x \in G_1, \\ f_2(x(t)) & \text{if } x \in G_2, \end{cases} \tag{3.1}$$

where

$$G_1 = \{x \in \mathbb{R}^2_+ \mid h(x) < 0\}, \qquad G_2 = \{x \in \mathbb{R}^2_+ \mid h(x) > 0\}.$$

Moreover, the switching manifold is defined by $\Sigma = \{x \in \mathbb{R}^2_+ \mid h(x) = 0\}$.

A distinctive feature of a Filippov-type system (3.1) is the possible evolution of trajectories along the switching manifold Σ, a phenomenon known as sliding motion. Specifically, trajectories that reach Σ are constrained to evolve within it, rather than transitioning immediately to another vector field. According to Filippov's convex method [37], the vector field governing the motion along the switching manifold is defined as a convex linear combination of the vector fields of the two subsystems; i.e.,

$$f_s(x) = (1 - \alpha(x))f_1(x) + \alpha(x)f_2(x) \tag{3.2}$$

with $\alpha(x) \in [0, 1]$, where

$$\alpha(x) = \frac{\nabla h(x) \cdot f_1(x)}{\nabla h(x) \cdot (f_1(x) - f_2(x))}.$$

Moreover, we can divide Σ into the following regions [37]:

(i) Sliding region: $\Sigma_s = \{x \in \Sigma \mid (\nabla h(x) \cdot f_1(x))(\nabla h(x) \cdot f_2(x)) \leqslant 0\}$;
(ii) Crossing region: $\Sigma_c = \{x \in \Sigma \mid (\nabla h(x) \cdot f_1(x))(\nabla h(x) \cdot f_2(x)) > 0\}$.

Sliding modes play a crucial role in the dynamics of Filippov systems. Mathematically, once a trajectory enters the sliding region, it evolves along this manifold rather than crossing it. Biologically, the sliding region corresponds to rapid alternations between the implementation and suspension of control and treatment measures, leading to shorter durations of both control and non-control phases. In contrast, a crossing region occurs when trajectories intersect the switching boundary transversally [38].

The equilibria x^* of system (3.1) are defined as follows [37]:

(i) Real equilibrium: $f_i(x^*) = 0$ and $x^* \in G_i$, where $i = 1, 2$;
(ii) Virtual equilibrium: $f_i(x^*) = 0$ and $x^* \in G_j$, where $i \neq j$ and $i, j = 1, 2$;
(iii) Pseudo-equilibrium: $f_s(x^*) = 0$ and $0 < \alpha(x^*) < 1$.

A real equilibrium is an equilibrium belonging to the region it lies in, which has not been excised. A virtual equilibrium is an equilibrium in a region that has been excised due the to Filippov definition, but which may still attract trajectories from another region. Both the real equilibrium and virtual equilibrium are called regular equilibria. By contrast, pseudo-equilibria are not equilibria of any individual region but are rather formed by the boundaries of Filippov regions [39].

3.2. The planar Filippov system (1.1) with $\tau > 0$

We now consider the planar Filippov system (1.1) incorporating a time delay in the switching manifold. When multiple switchings occur within a time interval of length τ, the system may exhibit highly complex dynamics. In this section, we restrict our attention to a relatively small time delay, such that the trajectory undergoes at most one switch during any interval of duration τ. Under these conditions, we will show that system (1.1) can admit a slowly oscillating periodic solution and that this solution is the unique limit cycle. To this end, we first provide an overview of the concept of a slowly oscillating periodic solution.

Definition 3.1. A solution $(x_1(t; x_0), x_2(t; x_0))$ of system (1.1) is called *slowly oscillating* on $[-\tau, +\infty)$ near Σ, if the following conditions are satisfied:

(i) there exists a time sequence $\{t_s\} \subseteq [-\tau, +\infty)$ such that each t_s is an isolated zero of $x_2(\cdot, x_0) - k$, with $t_s \to +\infty$ as $(s \to +\infty)$;
(ii) for any two adjacent zeros t_i and t_j of $x_2(\cdot, x_0) - k$, the separation satisfies $|t_i - t_j| \geqslant \tau$.

Furthermore, if such a solution is periodic, it is referred to as a slowly oscillating periodic solution of system (1.1).

Taking into account the time-shift invariance of autonomous equations and Definition 3.1, we choose the following state space:

$$\bar{\Omega} = \{x_0 \in \Omega \mid x_2(0) = k \text{ and } x_2(\theta) - k \neq 0 \text{ for all } \theta \in [-\tau, 0)\}.$$

We further define $\bar{\Omega}_1 = \{x_0 \in \bar{\Omega} \mid x_2(\theta) - k < 0\}, \bar{\Omega}_2 = \{x_0 \in \bar{\Omega} \mid x_2(\theta) - k > 0\}$, so that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$.

We next provide the definitions of the oscillating and crossing regions of system (1.1) with initial condition $x_0 \in \bar{\Omega}_i$, $i = 1$ or 2.

Definition 3.2. The *oscillating region* Σ_o^i of system (1.1) with initial condition $x_0 \in \bar{\Omega}_i$ is the subset of the discontinuity set for which

$$(\nabla h_i^\tau(\Phi_i(\tau, x)) \cdot f_1(\Phi_i(\tau, x)))(\nabla h_i^\tau(\Phi_i(\tau, x)) \cdot f_2(\Phi_i(\tau, x))) \leqslant 0,$$

where $x \in \Sigma$ and $h_i^\tau(\cdot) = 0$ denotes the smooth curve $\Phi_i(\tau, \{h(x) = 0\})$, with $i = 1$ or 2. If the above product is strictly positive, the corresponding subset is called *the crossing region* Σ_c^i. Furthermore, the oscillating and crossing regions of system (1.1) with initial condition $x_0 \in \bar{\Omega}$ are defined as $\Sigma_o = \Sigma_o^1 \cup \Sigma_o^2$ and $\Sigma_c = \Sigma_c^1 \cup \Sigma_c^2$, respectively.

Definition 3.3. The *oscillating space* M of system (1.1) is defined by

$$M = \{\Phi_1([0, \tau], P) \mid P \in \Sigma_o^1\} \cup \{\Phi_2([0, \tau], Q) \mid Q \in \Sigma_o^2\}. \tag{3.3}$$

Definition 3.4. For system (1.1), a point x^* is called an *admissible (real) equilibrium* if

$$f_i(x^*) = 0 \text{ and } x^* \in G_i \text{ where } i = 1, 2.$$

Alternatively, x^* is called a *virtual equilibrium* if

$$f_i(x^*) = 0 \text{ and } x^* \in G_j \text{ where } i \neq j \text{ and } i, j = 1, 2.$$

3.3. Existence and stability of limit cycles

Consider the planar Filippov system (3.1) with $h(x) = x_2 - k$, and assume it possesses a pseudo-equilibrium $E_s = (x_1^*, k)$. Applying the linear transformation

$$\begin{aligned} \bar{x}_1(t) &= x_1(t) - x_1^*, \\ \bar{x}_2(t) &= x_2(t) - k, \end{aligned} \tag{3.4}$$

the system is transformed (with the bars omitted for simplicity) into

$$x'(t) = \begin{cases} f_1(x(t)) & \text{if } x_2 < 0, \\ f_2(x(t)) & \text{if } x_2 > 0, \end{cases} \tag{3.5}$$

with the pseudo-equilibrium shifted to $E_s = (0, 0)$. The pseudo-equilibrium satisfies

$$(f_{12} f_{21} - f_{11} f_{22})|_{x=(0,0)} = 0, \tag{3.6}$$

and the vector field of the sliding motion along Σ_s is given by

$$x_1' = \frac{f_{12} f_{21} - f_{11} f_{22}}{f_{12} - f_{22}}, \qquad x_2 = 0.$$

Let

$$b_{01} = f_{12}(0, 0), b_{02} = f_{22}(0, 0),$$

and assume

$$b_{02} < 0 < b_{01} \tag{3.7}$$

such that the origin lies within an attracting sliding region. The stability of the origin is determined by the sign of

$$\rho = \frac{\partial (f_{12} f_{21} - f_{11} f_{22})}{\partial x_1}\Big|_{x=(0,0)}. \tag{3.8}$$

By constructing and analysing an appropriate Poincaré map, the properties of a slowly oscillating periodic solution can be derived. Without loss of generality, system (1.1) is assumed to have undergone the same transformation (3.4) as system (3.5).

Theorem 3.1. *Consider the planar Filippov system (1.1) with a relatively small time delay and initial condition $x_0 \in \bar{\Omega}$, where f_1 and f_2 are C^2. Assume that conditions (3.6) and (3.7) hold. Then there exists a unique stable (unstable) limit cycle in the oscillating space M for system (1.1) in a neighbourhood of $(x; \tau) = (0; 0)$ when $\rho < 0$ ($\rho > 0$). Moreover, the amplitude of this limit cycle is asymptotically proportional to τ, and its period is given by*

$$T = \left(2 - \frac{b_{01}}{b_{02}} - \frac{b_{02}}{b_{01}}\right) \tau + O(\tau^2).$$

Proof. We first describe the trajectory of system (1.1) passing through $x_2 = 0$. Let $p(t) \in \mathbb{R}^2$ denote the position of the trajectory at time t, with initial point $p(0) = (h, 0)$ for some $h \in \mathbb{R}$. Suppose that $p(t) \in G_j$ for $t \in (-\tau, 0)$ and $p(t) \notin G_j$ for sufficiently small $t > 0$, where $j = 1$ or 2. Then the trajectory undergoes the next switch at $p(\tau) = (\Phi_{j1}(\tau; h, 0), \Phi_{j2}(\tau; h, 0))$. By continuously varying h, the points $(\Phi_{11}(\tau; h, 0), \Phi_{12}(\tau; h, 0))$ and $(\Phi_{21}(\tau; h, 0), \Phi_{22}(\tau; h, 0))$ trace curves representing where trajectories of system (1.1) undergo their next switch (see the red lines in Fig. 1).

Next we construct a Poincaré map P. Let $P_U(h; \tau)$ denote the x_2-value of the first intersection of the forward solution of the upper subsystem $x'(t) = f_2(x(t))$, with initial condition $(\Phi_{11}(\tau; h, 0), \Phi_{12}(\tau; h, 0))$, satisfying $x_2 = 0$. Let $T_U(h; \tau)$ be the corresponding elapsed time. Then

$$\begin{aligned} P_U(h; \tau) &= \Phi_{21}\Big(T_U(h; \tau); \Phi_{11}(\tau; h, 0), \Phi_{12}(\tau; h, 0)\Big), \\ 0 &= \Phi_{22}\Big(T_U(h; \tau); \Phi_{11}(\tau; h, 0), \Phi_{12}(\tau; h, 0)\Big). \end{aligned} \tag{3.9}$$

Similarly, let $P_L(h; \tau)$ denote the x_2-value of the first intersection of the forward solution of the lower subsystem $x'(t) = f_1(x(t))$ with initial condition $(\Phi_{21}(\tau; h, 0), \Phi_{22}(\tau; h, 0))$, satisfying $x_2 = 0$, and let $T_L(h; \tau)$ be the corresponding elapsed time. Then

$$P = P_L \circ P_U, \quad T(h; \tau) = T_U(h; \tau) + T_L(P_U(h; \tau); \tau) + 2\tau.$$

Since f_1 and f_2 are C^2, we can expand them via Taylor series:

$$\begin{aligned} f_{j1}(x) &= a_{0j} + a_{1j}x_1 + a_{2j}x_2 + O\left((|x|+|y|)^2\right), \\ f_{j2}(x) &= b_{0j} + b_{1j}x_1 + b_{2j}x_2 + O\left((|x|+|y|)^2\right), \end{aligned} \tag{3.10}$$

where $j = 1, 2$. Conditions (3.6) and (3.8) imply

$$\begin{aligned} &b_{01}a_{02} = a_{01}b_{02}, \\ &\quad \rho = b_{11}a_{02} + b_{01}a_{12} - a_{11}b_{02} - a_{01}b_{12}. \end{aligned} \tag{3.11}$$

Using asymptotic matching, the orbit of the lower subsystem for small τ is

$$\begin{aligned} \Phi_{11}(\tau; h, 0) &= h + a_{01}\tau + a_{11}h\tau + \frac{1}{2}(a_{01}a_{11} + a_{21}b_{01})\tau^2 + o\left((|h|+|\tau|)^2\right), \\ \Phi_{12}(\tau; h, 0) &= b_{01}\tau + b_{11}h\tau + \frac{1}{2}(a_{01}b_{11} + b_{01}b_{21})\tau^2 + o\left((|h|+|\tau|)^2\right). \end{aligned} \tag{3.12}$$

Similarly, the upper subsystem yields $\Phi_{21}(t; x_1, x_2)$ and $\Phi_{22}(t; x_1, x_2)$. From (3.9), we obtain

$$\begin{aligned} P_U(h;\tau) = {} & \Phi_{11} - \frac{a_{02}}{b_{02}}\Phi_{12} + \frac{1}{2}\left(\frac{a_{22}}{b_{02}} + \frac{a_{02}(a_{12}-b_{22})}{b_{02}^2} - \frac{a_{02}^2 b_{12}}{b_{02}^3}\right)\Phi_{12}^2 \\ & + \frac{a_{02}b_{12} - a_{12}b_{02}}{b_{02}^2}\Phi_{11}\Phi_{12} + o((|h|+|\tau|)^2), \\ T_U(h;\tau) = {} & -\frac{1}{b_{02}}\Phi_{12} + \frac{b_{12}}{b_{02}^2}\Phi_{11}\Phi_{12} - \frac{a_{02}b_{12} + b_{02}b_{22}}{2b_{02}^3}\Phi_{12}^2 + o((|h|+|\tau|)^2). \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$P_U(h;\tau) = h + c_1 h\tau + c_2\tau^2 + o((|h|+|\tau|)^2), \quad T_U(h;\tau) = -\frac{b_{01}}{b_{02}}\tau + O((|h|+|\tau|)^2), \tag{3.14}$$

where

$$c_1 = \frac{b_{02}(a_{11}b_{02} - a_{02}b_{11}) + b_{01}(a_{02}b_{12} - a_{12}b_{02})}{b_{02}^2}$$

and $c_2 \in \mathbb{R}$ does not need to be solved specifically.

By symmetry,

$$P_L(h;\tau) = h + c_3 h\tau + c_4\tau^2 + o((|h|+|\tau|)^2), \quad T_L(h;\tau) = -\frac{b_{02}}{b_{01}}\tau + O((|h|+|\tau|)^2), \tag{3.15}$$

where

$$c_3 = \frac{b_{01}(a_{12}b_{01} - a_{01}b_{12}) + b_{02}(a_{01}b_{11} - a_{11}b_{01})}{b_{01}^2}$$

and $c_4 \in \mathbb{R}$ does not need to be solved specifically. It follows from (3.11) and (3.14)–(3.15) that

$$\begin{aligned} P(h;\tau) = {} & h + \frac{b_{02}-b_{01}}{b_{01}b_{02}}\rho h\tau + (c_3 + c_4)\tau^2 + o((|h|+|\tau|)^2), \\ T(h;\tau) = {} & \left(2 - \frac{b_{01}}{b_{02}} - \frac{b_{02}}{b_{01}}\right)\tau + O((|h|+|\tau|)^2), \end{aligned} \tag{3.16}$$

where ρ is defined in (3.11) and

$$\frac{b_{02}-b_{01}}{b_{01}b_{02}} > 0.$$

Define the displacement function

$$D(h;\tau) = P(h;\tau) - h.$$

To prove the existence of a unique limit cycle, we show that $D(h;\tau) = 0$ has a unique zero using the implicit function theorem. Since $P(h;\tau)$ is C^2 and $P(h;0) = h$, $D(h;\tau)$ is C^1 and can be extended to a neighbourhood of $(h;\tau) = (0;0)$. Then the implicit function theorem yields

$$h^*(\tau) = \frac{b_{01}b_{02}(c_3 + c_4)}{(b_{01} - b_{02})\rho}\tau + o(\tau), \tag{3.17}$$

which implies the existence of a limit cycle. From (3.17), (3.12), and the corresponding expressions for $\Phi_{2i}(\tau; h, 0)$, $i = 1, 2$, the amplitude of this limit cycle is asymptotically proportional to τ. Moreover,

$$\frac{\partial D(h;\tau)}{\partial h} = \frac{b_{02}-b_{01}}{b_{01}b_{02}}\rho\tau + o(1),$$

so the limit cycle is unstable for $\rho > 0$ and stable for $\rho < 0$. Its period is given by (3.16) evaluated at $h = h^*(\tau)$, and by definition of the oscillating space, the limit cycle lies within the oscillating space M. □

Remarks. 1. A specific range of values for the relatively small time delay will be provided in Section 4 for a system, ensuring that the trajectory is swapped between two successive switches in a time not less than τ.

2. In standard ODE systems, limit cycles are typically induced by a Hopf bifurcation; that is, the real part of the leading eigenvalue crosses the imaginary axis with nonzero speed. The limit cycle emerging from such an equilibrium has an amplitude asymptotically proportional to the square root of the change in the bifurcation parameter, and its period is $2\pi/\omega$, where $\pm i\omega$ are eigenvalues of the equilibrium. In contrast, the mechanism generating the limit cycle in Theorem 3.1 differs from that of classical ODEs. Here, the amplitude is asymptotically proportional to the change in the bifurcation parameter, and the period is T. Notably, the oscillating solution is primarily driven by the delayed switching mechanism; i.e., the alternation between the two subsystems, rather than by the variation of a parameter.

3. When $\tau > 0$, system (1.1) with initial condition $x_0 \in \bar{\Omega}$ exhibits oscillatory motion rather than sliding motion. In this case, there is no pseudo-equilibrium in the sliding region; instead, a limit cycle appears in the oscillating space M. Furthermore, whenever a pseudo-equilibrium exists in the classical Filippov system (3.1), a corresponding limit cycle exists in system (1), and both share the same stability properties. Moreover, as $\tau \to 0$, Definition 3.2 reduces to the sliding region of the classical Filippov system.

4. Application to the SIR model

In this section, we apply system (1.1) to an epidemiological model to justify our theoretical results. For completeness of dynamical analysis, we also examine the global behaviour of the system with respect to its real equilibria.

4.1. Model development and preliminary analysis

Control measures may be implemented or suspended once the number of infected individuals reaches a certain critical threshold [5,23]. Biologically, if the threshold strategy and other model parameters are chosen appropriately, it is possible to maintain the number of infected individuals at a desired target level.

To analyse the impact of a delayed threshold policy, we consider a population divided into three disjoint compartments: susceptible (S), infectious (I) and recovered (R). We assume that all newborns are susceptible and that the birth and death rates are the same across all compartments, so that the total population $N = S + I + R$ remains constant. For convenience, we scale the population such that $N = 1$. We further assume that the number of infected individuals at time $t-\tau$ determines whether the control policy is applied at time t, where τ represents the response delay. Based on these assumptions, we propose the following Filippov SIR model with time delay in the switching decision:

$$\begin{aligned} \frac{dS}{dt} &= \mu - \beta \exp(-\alpha\sigma I)SI - \mu S, \\ \frac{dI}{dt} &= \beta \exp(-\alpha\sigma I)SI - (\mu+\gamma)I, \\ \frac{dR}{dt} &= \gamma I - \mu R, \end{aligned} \tag{4.1}$$

with

$$\sigma(I) = \begin{cases} 0, & I(t-\tau) < k, \\ 1, & I(t-\tau) > k. \end{cases} \tag{4.2}$$

Here $\mu > 0, \beta > 0$, and $\gamma > 0$ denote the birth/death rate, infection rate and recovery rate, respectively. The parameter $\alpha > 0$ represents the influence of media coverage on disease transmission, and τ accounts for the delay in reporting. The threshold $k \in (0,1)$ specifies when control measures are implemented, meaning that interventions at time t are triggered once the number of infected individuals I reaches k at time $t-\tau$. It is worth noting that the dynamics of system (4.1)–(4.2) with $\tau = 0$ have been fully analysed in [23].

For consistency with the notation used throughout this paper, let $x_1 = S$ and $x_2 = I$. Since R does not influence the first two equations of system (4.1), it suffices to analyse the reduced model:

$$\begin{aligned} \frac{dx_1}{dt} &= \mu - \beta \exp(-\alpha\sigma x_2)x_1x_2 - \mu x_1, \\ \frac{dx_2}{dt} &= \beta \exp(-\alpha\sigma x_2)x_1x_2 - (\mu+\gamma)x_2, \end{aligned} \tag{4.3}$$

where

$$\sigma = \begin{cases} 0, & x_2(t-\tau) < k, \\ 1, & x_2(t-\tau) > k. \end{cases} \tag{4.4}$$

System (4.3) consists of the free subsystem (lower subsystem)

$$x'(t) = \begin{pmatrix} \mu - \beta x_1x_2 - \mu x_1 \\ \beta x_1x_2 - (\mu+\gamma)x_2 \end{pmatrix} \equiv f_1(x) \tag{4.5}$$

and the control subsystem (upper subsystem)

$$x'(t) = \begin{pmatrix} \mu - \beta \exp(-\alpha x_2) x_1 x_2 - \mu x_1 \\ \beta \exp(-\alpha x_2) x_1 x_2 - (\mu+\gamma) x_2 \end{pmatrix} \equiv f_2(x) \tag{4.6}$$

The switching manifold is defined as

$$\Sigma = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_2 = k\},$$

which divides $\mathbb{R}_+^2$ into two regions

$$G_1 = \{(x_1, x_2) \in R_+^2 \mid x_2 < k\}, \quad G_2 = \{(x_1, x_2) \in R_+^2 \mid x_2 > k\}.$$

The basic reproduction number, equilibria and global stability of the subsystems corresponding to model (4.1) are well established in [23,40] and can be summarized as follows. Both subsystems share the same disease-free equilibrium $E_0 = (1, 0)$ and the same basic reproduction number $\mathcal{R}_0 = \beta/(\mu+\gamma)$. Moreover, the free subsystem (4.5) and the control subsystem (4.6) each admit an endemic equilibrium, denoted by $E_1 = (x_1^f, x_2^f)$ and $E_2 = (x_1^c, x_2^c)$, respectively, where

$$x_1^f = \frac{\mu+\gamma}{\beta}, \qquad x_2^f = \frac{\beta\mu - \mu(\mu+\gamma)}{\beta(\mu+\gamma)},$$
$$x_1^c = \frac{\mu+\gamma}{\beta}\exp(\alpha x_2^c), \qquad x_2^c = \frac{\mu}{\mu+\gamma} - \frac{1}{\alpha}\,\text{Lambert } W\left(\frac{\alpha\mu}{\beta}\exp(\frac{\alpha\mu}{\mu+\gamma})\right),$$

(see [41] for the definition of the Lambert W function used to express these equilibria). Since the components of both E_1 and E_2 satisfy the relation $\mu - \mu x_1 - (\mu+\gamma)x_2 = 0$ and because $x_1^f < x_1^c$, it follows that $x_1^* + x_2^* \leqslant 1$ and $x_2^f > x_2^c$, where $* \in \{f, c\}$.

Lemma 4.1.

(i) For the free subsystem (4.5), if $\mathcal{R}_0 \leqslant 1$, E_0 is globally asymptotically stable; if $\mathcal{R}_0 > 1$, E_1 is globally asymptotically stable.
(ii) For the control subsystem (4.6), if $\mathcal{R}_0 \leqslant 1$, E_0 is globally asymptotically stable; if $\mathcal{R}_0 > 1$, E_2 is globally asymptotically stable.

The initial state space is defined as

$$\tilde{\Omega} = \{(x_{10}, x_2(\theta)) \in \Omega \mid 0 < x_2(0) \text{ and } x_{10} + x_2(0) \leqslant 1\},$$

where $C = C([-\tau, 0], [0, 1])$. By Theorems 2.1–2.3, system (4.3)–(4.4) with initial condition $x_0 \in \tilde{\Omega}$ admits a unique positive solution on $[-\tau, +\infty)$.

Next, we show that $\tilde{\Omega}$ is a positive invariant set for system (4.3)–(4.4). For this, we first establish the following result.

Proposition 4.1. *For $k \in (0, 1)$, the following assertions hold:*

(i) If $k \neq x_2^f$, then $\Phi_{12}(t; x_1^{f0}, x_2^{f0}) - k$ has only finitely many zeros in any bounded interval of $[0, +\infty)$, where $(\Phi_{11}(t; x_1^{f0}, x_2^{f0}), \Phi_{12}(t; x_1^{f0}, x_2^{f0}))$ is any nontrivial solution of the free subsystem (4.5) with initial condition (x_1^{f0}, x_2^{f0}), $x_i^{f0} = x_i^f(0)$, $i = 1, 2$;
(ii) If $k \neq x_2^c$, then $\Phi_{22}(t; x_1^{c0}, x_2^{c0}) - k$ has only finitely many zeros in any bounded interval of $[0, +\infty)$, where $(\Phi_{21}(t; x_1^{c0}, x_2^{c0}), \Phi_{22}(t; x_1^{c0}, x_2^{c0}))$ is any nontrivial solution of the control subsystem (4.6) with initial condition (x_1^{c0}, x_2^{c0}), $x_i^{c0} = x_i^c(0)$, $i = 1, 2$.

Proof. We present the proof for case (i). Let $g(t) = \Phi_{12}(t; x_1^{f0}, x_2^{f0}) - k$. We claim that any zero t^* of $g(t)$ is isolated; i.e., there exists $\epsilon > 0$ sufficiently small such that $g(t^* - \epsilon)g(t^* + \epsilon) < 0$. Indeed, if $\Phi_{12}(t^*; x_1^{f0}, x_2^{f0}) \neq x_1^f$, then

$$g'(t^*) = \beta k[\Phi_{12}(t^*; x_1^{f0}, x_2^{f0}) - x_1^f] \neq 0.$$

If $\Phi_{12}(t^*; x_1^{f0}, x_2^{f0}) = x_1^f$, then $g'(t^*) = 0$ but

$$g''(t^*) = \beta^2 x_1^f k(x_2^f - k) \neq 0.$$

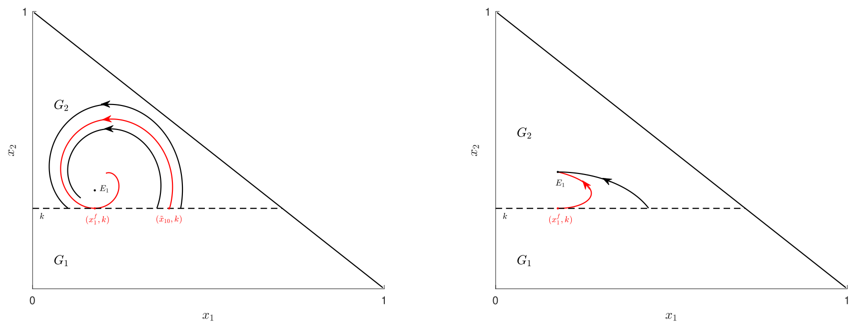
In either case, t^* is an isolated zero of $g(t)$. Therefore, $g(t)$ has only finitely many zeros in any bounded interval of $[0, +\infty)$. Case (ii) can be proved similarly. □

Proposition 4.2. *If $x_0 \in \tilde{\Omega}$, then system (4.3)–(4.4) has a unique solution $(x_1(t), x_{2,t}(\theta)) \in \tilde{\Omega}$ passing through x_0 for all $t \in [0, +\infty)$, where $x_{2,t}(\theta) = x_2(t + \theta)$.*

Proof. Let $N(t) = x_1(t) + x_{2,t}(0)$. It is straightforward to verify that

$$N'(t) \leqslant \mu - \mu N(t).$$

Hence, for $x_0 \in \tilde{\Omega}$ and $t \geqslant 0$, we have $N(t) \leqslant 1$. Moreover, by combining Proposition 4.1 with the solution construction (2.1)–(2.3), we conclude that $\tilde{\Omega}$ is a positive invariant under system (4.3)–(4.4). That is, $(x_1(t), x_{2,t}(\theta)) \in \tilde{\Omega}$ for all $t \in [0, +\infty)$. □



(a) $0 < T_f < +\infty$ (b) $T_f = +\infty$

Fig. 2. Schematic diagrams illustrating two distinct cases for the value of T_f.

4.2. Bifurcation analysis

We now apply the theory developed in Section 3.3 to determine a suitable range for the relatively small time delay.

For system (4.3)–(4.4) with $\tau = 0$, the sliding-mode dynamics (3.2) are governed by

$$x_1' = \mu - \mu x_1 - (\mu + \gamma)k, \qquad x_2 = k. \tag{4.7}$$

The corresponding sliding region is defined as

$$\Sigma_s = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 \;\middle|\; \frac{\mu + \gamma}{\beta} \leqslant x_1 \leqslant \frac{(\mu+\gamma)\exp(\alpha k)}{\beta}, x_2 = k \right\}.$$

There exists a pseudo-equilibrium $E_s = (x_1^*, x_2^*) = ((\mu - (\mu + \gamma)k)/\mu, k)$ for system (4.3)–(4.4) with $\tau = 0$ provided that

$$\frac{\mu + \gamma}{\beta} < x_1^* < \frac{(\mu+\gamma)\exp(\alpha k)}{\beta},$$

which is equivalent to $x_2^c < k < x_2^f$. Hence, conditions (3.6) and (3.7) of Theorem 3.1 are satisfied whenever $x_2^c < k < x_2^f$.

Next, for the epidemiological model (4.3)–(4.4), we identify a range of small time delays that satisfy Theorem 3.1, focusing on the case

$$x_2^c < k < x_2^f.$$

To determine this range, we perform a phase-plane analysis of system (4.3)–(4.4).

For the free subsystem (4.5) of system (4.3)–(4.4) under delayed interventions, it may occur that there exists $\tilde{x}_{10} \in [x_1^f, x_{f0})$ such that the solution starting from $(\tilde{x}_{10}, k)$ reaches the point (x_1^f, k) in finite time, where $x_{f0} = 1 - k$, and (x_1^f, k) is the intersection of the x_2-nullcline of system (4.5) with the line $x_2 = k$. If such a $\tilde{x}_{10}$ exists, then all solutions starting from the set $\{(x_1, x_2) \mid \tilde{x}_{10} \leqslant x_1 \leqslant x_{f0}, x_2 = k\}$ will enter region G_2. Furthermore, these solutions will return to the line Σ in a finite time $\tilde{t} = \tilde{t}(x_1)$. By Lemma 4.1, solutions starting from $\{(x_1, x_2) \mid x_1^f \leqslant x_1 < \tilde{x}_{10}, I = k\}$ cannot return to Σ; in this case, we set $\tilde{t}(x_1) = +\infty$. Define

$$T_f = \min_{x_1 \in [\tilde{x}_{10}, x_{f0}]} \tilde{t}(x_1). \tag{4.8}$$

Then $0 < T_f < +\infty$ if $\tilde{x}_{10}$ exists (see Fig. 2(a)). Otherwise, if no such $\tilde{x}_{10}$ exists, we set $T_f = +\infty$ (see Fig. 2(b)).

For the control subsystem (4.6) of system (4.3)–(4.4) under delayed interventions, it may occur that there exists $\bar{x}_{10} \in [0, x_{c0})$ such that the solution starting from $(\bar{x}_{10}, k)$ reaches the point (x_{c0}, k) in finite time, where $x_{c0} = (\mu + \gamma)\exp(\alpha k)/\beta$ and (x_{c0}, k) is the intersection of the x_2-nullcline of system (4.6) with the line $x_2 = k$. In this case, all solutions starting from $\{(x_1, x_2) \mid 0 \leqslant x_1 \leqslant \bar{x}_{10}, x_2 = k\}$ will enter region G_1. Furthermore, these solutions will return to the line Σ in a finite time $\bar{t} = \bar{t}(x_1)$. By Lemma 4.1, solutions starting from $\{(x_1, x_2) \mid \bar{x}_{10} < x_1 \leqslant x_{c0}, x_2 = k\}$ cannot return to Σ; in this case, we set $\bar{t}(x_1) = +\infty$ for convenience. Define

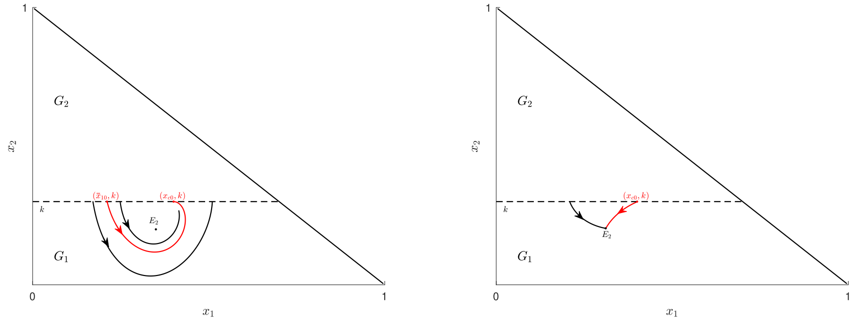
$$T_c = \min_{x_1 \in [0, x_{c0}]} \bar{t}(x_1). \tag{4.9}$$

If such a $\bar{x}_{10}$ exists, then $0 < T_c < +\infty$ (see Fig. 3(a)). Otherwise, solutions starting from $\{(x_1, x_2) \mid 0 \leqslant x_1 \leqslant x_{c0}, I = k\}$ cannot return to Σ, and we set $T_c = +\infty$ (see Fig. 3(b)).

Finally, let

$$T_0 = \min\{T_f, T_c\}. \tag{4.10}$$

It follows that any time delay satisfying $0 < \tau < T_0$ corresponds precisely to the relatively small time delay required for the system.



(a) $0 < T_c < +\infty$ (b) $T_c = +\infty$

Fig. 3. Schematic diagrams illustrating two distinct cases for the value of T_c.

Theorem 4.1. *If $x_2^c < k < x_2^f$ and $0 < \tau < T_0$, then system (4.3)–(4.4) admits a slowly oscillating solution $\hat{x}(t) = x_1(t; x_{10}, x_2(\theta)), x_2(t; x_{10}, x_2(\theta))$ with initial condition $x_0 \in \bar{\Omega}$.*

Proof. We first claim that $x_2(t; x_{10}, x_2(\theta))$ is oscillatory on $[-\tau, +\infty)$ around the switching manifold Σ. Suppose otherwise; that is, the zeros $\{t_n\}$ of $x_2(\cdot; x_{10}, x_2(\theta)) - k$ are bounded. Without loss of generality, assume that $x_2(t; x_{10}, x_2(\theta)) - k < 0$ for all $t > t_m$, where $t_m = \max\{t_n\}$. Then, for $t \geqslant t_m + \tau$, $x_2(t; x_{10}, x_2(\theta))$ satisfies the free system (4.5). By Lemma 4.1, it follows that $\lim_{t\to+\infty} x_2(t; x_{10}, x_2(\theta)) = x_2^f$, which contradicts the assumption $x_2^c < k < x_2^f$.

Next, we show that $x_2(t; x_{10}, x_2(\theta))$ is slowly oscillating on $[-\tau, +\infty)$. Since the system is autonomous, its solutions are invariant under time shifts. Without loss of generality, let $t_0 = 0$ and $t_1 > 0$ be two consecutive zeros of $x_2(t; x_{10}, x_2(\theta)) - k = 0$, with the initial history satisfying $x_2(\theta) < k$ for $\theta \in [-\tau, 0)$ and $x_2(0) = k$. Then existence of a slow oscillation solution is equivalent to showing that $t_1 > \tau$. For $t \in [0, \tau]$, the solution of system (4.3)–(4.4) coincides with that of the free subsystem (4.5) for $t \in [0, \tau]$, namely,

$$\begin{cases} x_1(t; x_{10}, x_2(\theta)) = \Phi_{11}(t; x_{10}, k), \\ x_2(t; x_{10}, x_2(\theta)) = \Phi_{12}(t; x_{10}, k), \end{cases} \quad t \in [0, \tau].$$

Since $\tau < T_f$, it follows that $x_2(t; x_{10}, x_2(\theta)) > k$ for $(0, \tau]$. By Lemma 4.1 and the condition $k \in (x_2^c, x_2^f)$, there exists $\bar{t}_1 > 0$ such that

$$x_2(\bar{t}_1; \Phi_{11}(\tau; x_{10}, k), \Phi_{12}(\tau; x_{10}, k)) = k$$

and

$$x_2(t; \Phi_{11}(\tau; x_{10}, k), \Phi_{12}(\tau; x_{10}, k)) > k \quad \text{for} \quad t \in [0, \bar{t}_1).$$

Therefore, for $t \in [\tau, \bar{t}_1 + \tau]$, the solution of system (4.3)–(4.4) coincides with that of the control subsystem (4.6); i.e.,

$$\begin{cases} x_1(t; x_{10}, x_2(\theta)) = \Phi_{21}(t - \tau; \Phi_{11}(\tau; x_{10}, k), \Phi_{12}(\tau; x_{10}, k)), \\ x_2(t; x_{10}, x_2(\theta)) = \Phi_{22}(t - \tau; \Phi_{11}(\tau; x_{10}, k), \Phi_{12}(\tau; x_{10}, k)), \end{cases} \quad t \in [\tau, \bar{t}_1 + \tau].$$

Consequently, $x_2(\bar{t}_1 + \tau; x_{10}, k) = \Phi_{22}(\bar{t}_1; \Phi_{11}(\tau; x_{10}, k), \Phi_{12}(\tau; x_{10}, k)) = k$, and hence $t_1 = \bar{t}_1 + \tau > \tau$. This establishes that the solution $x_1(t; x_{10}, x_2(\theta)), x_2(t; x_{10}, x_2(\theta))$ of system (4.3)–(4.4) is slowly oscillating. Therefore, when $\tau \in (0, T_0)$, the time delay τ is precisely the relatively small delay for which two successive switches of the trajectory occur at intervals no shorter than τ. □

It follows from (3.8) and (4.7) that $\rho = -\mu < 0$. Note that the linear transformation (3.4) represents a coordinate shift. In combination with Theorem 3.1, we then obtain the following result.

Theorem 4.2. *Consider system (4.3)–(4.4) with initial data $x_0 \in \bar{\Omega}$, if $k \in (x_2^c, x_2^f)$ and $\tau \in (0, T_0)$, then the system admits a unique stable limit cycle in the oscillatory domain M, located in a neighbourhood of $(x; \tau) = (x_1^*, k; 0)$. Moreover, the amplitude of this limit cycle is asymptotically proportional to τ, and its period is given by*

$$\tilde{T} = \left(2 - \frac{\tilde{b_{01}}}{\tilde{b_{02}}} - \frac{\tilde{b_{02}}}{\tilde{b_{01}}}\right)\tau + O(\tau^2),$$

where $\tilde{b_{01}} = f_{12}(x_1^*, k), \tilde{b_{02}} = f_{22}(x_1^*, k)$.

Remarks. Theorem 4.2 shows that system (4.3)–(4.4) admits periodic solutions under certain conditions, reflecting cyclical outbreaks and declines of the disease within a population. The amplitude and period of these solutions highlight the critical role of timely interventions: a smaller amplitude corresponds to lower epidemic peaks, while a shorter period indicates a narrower window for implementing effective control measures.

4.3. Global stability

After analysing the oscillatory behaviour near the switching manifold under certain conditions, it is natural to further examine the global dynamics of system (4.3)–(4.4). In particular, beyond the case $\mathcal{R}_0 > 1$ and $x_2^c < k < x_2^f$ discussed in Section 4.2, we are interested in establishing the global asymptotic stability and persistence properties of the system.

Theorem 4.3. *If $\mathcal{R}_0 \leqslant 1$, then the disease-free equilibrium $E_0 = (1,0)$ of system (4.3)–(4.4) is globally asymptotically stable. If $\mathcal{R}_0 > 1$, then the disease is uniformly persistent.*

Proof. System (4.3)–(4.4) can be rewritten in the form

$$\begin{aligned} x_1'(t) &= -\left(\beta\exp(-\alpha\sigma x_2)x_2+\mu\right)(x_1-1)-\beta\exp(-\alpha\sigma x_2)x_2,\\ x_2'(t) &= \beta\exp(-\alpha\sigma x_2)(x_1-1)x_2-(\mu+\gamma)\left(1-\mathcal{R}_0\exp(-\alpha\sigma x_2)\right)x_2, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} &\sigma = 0, && x_2(t-\tau) < k,\\ &\sigma \in \{0,1\}, && x_2(t-\tau) = k,\\ &\sigma = 1, && x_2(t-\tau) > k. \end{aligned}$$

Consider the Lyapunov function

$$V(x_1,x_2) = \frac{1}{2}(x_1-1)^2 + x_2.$$

Differentiating $V(x(t))$ along trajectories of system (4.11) yields

$$\begin{aligned} \frac{dV}{dt}\Big|_{(4.11)} &= -\left(\beta\exp(-\alpha\sigma x_2)x_2+\mu\right)(x_1-1)^2-(\mu+\gamma)\left(1-\mathcal{R}_0\exp(-\alpha\sigma x_2)\right)x_2\\ &< -\left(\beta\exp(-\alpha\sigma x_2)x_2+\mu\right)(x_1-1)^2 \leqslant 0 \end{aligned}$$

whenever $\mathcal{R}_0 \leqslant 1$. Obviously, $V'(t) = 0$ if and only if $x_1 = 1$. From $x_1'(t) = 0$, it follows that $x_2(t) = 0$. Hence, the singleton $\{E_0\}$ is the largest compact invariant set contained in $\{V'(t) = 0\}$. By LaSalle's invariance principle, E_0 is globally asymptotically stable when $\mathcal{R}_0 \leqslant 1$.

We now show that the disease is uniformly persistent when $\mathcal{R}_0 > 1$. Consider a solution of system (4.3)–(4.4) with $x_2(0) > 0$. If there is a sufficiently large t_1 such that $x_2(t_1) \geqslant k$, then $\limsup_{t\to+\infty} x_2(t) \geqslant k$. Otherwise, there exists t_2 such that $x_2(t) < k$, $t \in (t_2,+\infty)$. In this case, the solution coincides with that of the free subsystem (4.5) for $t \in [t_2+\tau,+\infty)$; that is,

$$\begin{cases} x_1(t) = \Phi_{11}(t-t_2-\tau; x_1(t_2+\tau), x_2(t_2+\tau)),\\ x_2(t) = \Phi_{12}(t-t_2-\tau; x_1(t_2+\tau), x_2(t_2+\tau)), \end{cases} \qquad t \in [t_2+\tau,+\infty),$$

and hence $\lim_{t\to+\infty} x_2(t) = x_2^f$. In either case, we have $\limsup_{t\to+\infty} x_2(t) \geqslant \min\{k, x_2^f\}$, which implies uniform weak persistence. Since both subsystems of system (4.3) possess uniformly bounded derivatives on the set $\{(x_1,x_2) \in [0,1]\times[0,1] \mid x_1+x_2 \leqslant 1\}$, the solution operator of system (4.3) is compact for $t > \tau$ by the Arzelà–Ascoli theorem. Therefore, the system admits a compact attractor in $\hat{\Omega}$. By Corollary 4.8 in [42], the disease is uniformly persistent; that is, there exists $\epsilon > 0$ such that $\liminf_{t\to+\infty} x_2(t) \geqslant \epsilon$. □

Remarks. Theorem 4.3 shows that if the basic reproduction number satisfies $\mathcal{R}_0 \leqslant 1$, each infected individual produces, on average, at most one secondary infection, so the disease cannot sustain itself in the population [43]. Consequently, the system converges to the disease-free equilibrium $E_0 = (1,0)$, meaning the infection will eventually die out. Conversely, if $\mathcal{R}_0 > 1$, each infected individual generates more than one secondary infection on average, allowing the disease to persist in the population. In this case, the infection remains present over time; i.e., the disease is uniformly persistent.

Theorem 4.4. *If $\mathcal{R}_0 > 1$ and $\bar{k} < k$, where $\bar{k} = 1-(\mu+\gamma)/\beta$, then the endemic equilibrium E_1 of system (4.3)–(4.4) is globally asymptotically stable.*

Proof. It is straightforward to verify that $k > x_2^f$ holds when $k > \bar{k}$. Hence, in a small neighbourhood of E_1, the dynamics of system (4.3) coincide with those of system (4.5). According to Lemma 4.1, E_1^* is locally asymptotically stable for system (4.3). We next show that its domain of attraction is $\hat{\Omega}$ in three steps.

We first claim that there exists $t_0 \geqslant 0$ such that $x_2(t_0) < k$. If not, $x_2(t) \geqslant k$ for all $t \in [0,+\infty)$. Consequently,

$$x_2(t) = \Phi_{22}(t-\tau; x_1(\tau), x_2(\tau)) \quad \text{for} \quad t \in [\tau,+\infty).$$

Since $\mathcal{R}_0 > 1$, Lemma 4.1 implies that E_2 attracts all trajectories in $\hat{\Omega}$; i.e., $x_2(t) \to x_2^c$ as $t \to +\infty$. However, this contradicts $x_2^c < k$. Therefore, the claim holds.

Next, we claim that if $x_2(t_0) < k$ for some $t_0 \geqslant 0$, then $x_2(t) < k$ for all $t \in [t_0,+\infty)$. If not, there exists $t_1 > t_0$ such that $x_2(t_1) = k$ and $x_2(t) < k$ for $t \in [t_0,t_1)$ and $x_2'(t_1) \geqslant 0$. On the other hand,

$$x_1(t_1) \leqslant 1 - x_2(t_1) < 1 - \bar{k} = \frac{\mu+\gamma}{\beta},$$

and thus

$$x_2'(t_1) = \left[\beta \exp(-\alpha\sigma x_2(t_1))x_1(t_1) - (\mu+\gamma)\right] x_2(t_1) < 0.$$

This is in contradiction with $x_2'(t_1) \geqslant 0$, so the claim is true.

Finally, we claim that $(x_1(t), x_2(t)) \to E_1$ as $t \to +\infty$. The solution of system (4.3) coincides with that of system (4.5) for $t \in [t_0+\tau, +\infty)$:

$$x_1(t) = \Phi_{11}(t-(t_0+\tau); x_1(t_0+\tau), x_2(t_0+\tau)),$$
$$x_2(t) = \Phi_{12}(t-(t_0+\tau); x_1(t_0+\tau), x_2(t_0+\tau)).$$

We obtain that $(x_1(t), x_2(t)) \to E_1$ as $t \to +\infty$ by Lemma 4.1.

In summary, E_1 is locally asymptotically stable and attracts $\hat{\Omega}$ for system (4.3). Therefore, E_1 is globally asymptotically stable whenever $\mathcal{R}_0 > 1$ and $k > \bar{k}$. □

Remarks. Theorem 4.4 indicates that when $\mathcal{R}_0 > 1$ (so the disease can potentially spread) and the control threshold k is higher than the critical level $\bar{k}$, the disease persists in the population at the endemic equilibrium E_1. Biologically, this means that if interventions are implemented only when the number of infected individuals exceeds a relatively high threshold, the disease will stabilize at a persistent endemic level rather than being eradicated.

We now present the following lemma, which will be used to establish the global asymptotic stability of E_2. Decompose the state plane $\Delta \equiv \{(x_1,x_2) \in (0,1]\times(0,1] \mid x_1+x_2 \leqslant 1\}$ into two regions, $\Delta = \Delta_1 \cup \Delta_2$, where

$$\Delta_1 = \left\{ (x_1,x_2) \in \Delta \;\middle|\; x_1+x_2 \geqslant \frac{\mu}{\mu+\gamma} \right\},$$
$$\Delta_2 = \left\{ (x_1,x_2) \in \Delta \;\middle|\; x_1+x_2 < \frac{\mu}{\mu+\gamma} \right\}.$$

Lemma 4.2. *Assuming that the initial data of system (4.3) satisfies $(x_{10}, x_2(0)) \in \Delta_2$ and*

$$2\sqrt{\frac{\mu}{\beta}} > \frac{\mu}{\mu+\gamma} + \frac{\mu}{\beta}, \tag{4.12}$$

then there exists $t_0 > 0$ such that $(x_1(t_0; x_{10}, x_2(\theta)), x_2(t_0; x_{10}, x_2(\theta))) \in \Delta_1$, and moreover, $(x_1(t; x_{10}, x_2(\theta)), x_2(t; x_{10}, x_2(\theta))) \in \Delta_1$ for any $t \in [t_0, +\infty)$.

Proof. We first show that both subsystems (4.6)–(4.7) satisfy $f_{11}(x_1,x_2) > 0$ and $f_{21}(x_1,x_2) > 0$ in the closed region $\overline{\Delta_2}$. For any $x_1 \in (0,1]$, condition (4.12) implies

$$x_1 + \frac{\mu}{\beta x_1} \geqslant 2\sqrt{\frac{\mu}{\beta}} > \frac{\mu}{\mu+\gamma} + \frac{\mu}{\beta}.$$

Therefore,

$$\frac{\mu - \mu x_1}{\beta x_1} > \frac{\mu}{\mu+\gamma} - x_1 \quad \text{for} \quad x_1 \in \left(0, \frac{\mu}{\mu+\gamma}\right].$$

From the vector fields of subsystems (4.6)–(4.7), it follows that $f_{11}(x_1,x_2) > 0$ and $f_{21}(x_1,x_2) > 0$ in

$$\left\{ (x_1,x_2) \in \Delta \;\middle|\; x_2 < \frac{\mu - \mu x_1}{\beta x_1} \right\}.$$

Hence, both subsystems satisfy $f_{11}(x_1,x_2) > 0$ and $f_{21}(x_1,x_2) > 0$ throughout $\overline{\Delta_2}$.

Consider now the trajectory $(x_1(t; x_{10}, x_2(\theta)), x_2(t; x_{10}, x_2(\theta)))$ of system (4.3) with $(x_{10}, x_2(0)) \in \Delta_2$. Since $x_1'(t; x_{10}, x_2(\theta)) > 0$ for trajectories in Δ_2 and Δ_2 is bounded, the solution must eventually exit Δ_2 at some time t_0. This is, $(x_1(t_0; x_{10}, x_2(\theta)), x_2(t_0; x_{10}, x_2(\theta))) \in \Delta_1$.

We next show that the solution remains in Δ_1 thereafter. For $t \geqslant t_0$, system (4.3) yields

$$(x_1(t) + x_2(t))' \geqslant \mu - (\mu+\gamma)(x_1(t)+x_2(t)).$$

Consider the auxiliary system:

$$v'(t) = \mu - (\mu+\gamma)v(t),$$
$$v(t_0) = v_0 \geqslant \frac{\mu}{\mu+\gamma}.$$

By the comparison principle,

$$x_1(t; x_{10}, x_2(\theta)) + x_2(t; x_{10}, x_2(\theta)) \geqslant v(t) \geqslant \frac{\mu}{\mu+\gamma}$$

for $t \in [t_0, +\infty)$, which implies $(x_1(t; x_{10}, x_2(\theta)), x_2(t; x_{10}, x_2(\theta))) \in \Delta_1$ for any $t \in [t_0, +\infty)$. This completes the proof. □

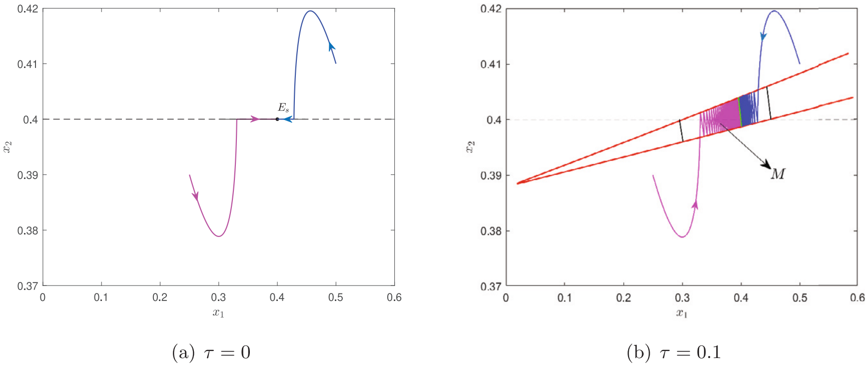


Fig. 4. Illustration of the difference between Filippov systems with and without delay. (*a*) The pseudo-equilibrium $E_s = (0.4, 0.4)$ of system (4.3) with $\tau = 0$ is globally asymptotically stable. (*b*) A unique and stable limit cycle (shown in green) of system (4.3) with $\tau = 0.1$ exists in the oscillating space M, where M is a curved quadrilateral bounded by the red and black lines. The parameters used are $\mu = 0.2, \beta = 1, \alpha = 1, \gamma = 0.1$ and $k = 0.4$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Theorem 4.5. *Assume that $\mathcal{R}_0 > 1$, condition (4.12) holds, and $0 < k < \tilde{k}$, where $\tilde{k} = \mu/(\mu+\gamma) - x_1^c$. Then endemic equilibrium E_2 of system (4.3) is globally asymptotically stable.*

Proof. Since $0 < k < \tilde{k}$, it follows that $k < x_2^c < x_2^f$. By Lemma 4.1, the equilibrium E_2 is locally asymptotically stable for system (4.3). We only need to show that $(x_1(t; x_{10}, x_2(\theta)), x_2(t; x_{10}, x_2(\theta))) \to E_2$ as $t \to +\infty$, for any initial condition $(x_{10}, x_2(\theta)) \in \hat{\Omega}$. By Lemma 4.2, there exists t_0 such that $(x_1(t; x_{10}, x_2(\theta)), x_2(t; x_{10}, x_2(\theta))) \in \Delta_1$ for any $t \in [t_0, +\infty)$.

We first show that there exists $t_1 \geqslant t_0$ such that $x_2(t_1; x_{10}, x_2(\theta)) \geqslant k$. Otherwise, $x_2(t_1; x_{10}, x_2(\theta)) < k$ for any $t \in [t_0, +\infty)$. For $t > t_0 + \tau$, we obtain

$$x_2(t) = \Phi_{12}(t - t_0 - \tau; x_1(t_0 + \tau), x_2(t_0 + \tau)).$$

It follows from Lemma 4.1 that $x_2(t_1; x_{10}, x_2(\theta)) \to x_2^f$ as $t \to +\infty$, which contradicts the fact that $k < x_2^f$. Hence, the claim holds.

Next, we show that $x_2(t) \geqslant k$ for all $t \in [t_1, +\infty)$. Suppose, on the contrary, that there exists $t_2 \in (t_1, +\infty)$ such that $x_2(t_2) = k$, and $x_2'(t_2) \leqslant 0$. Since $(x_1(t_2), x_2(t_2)) \in \Delta_1$ by Lemma 4.2, we have

$$x_1(t_2) \geqslant \frac{\mu}{\mu+\gamma} - k > x_1^c.$$

Then,

$$x_2'(t_2) > [\beta \exp(-\alpha k) x_1^c - (\mu+\gamma)]k > 0,$$

which contradicts $x_2'(t_2) \leqslant 0$. This means that the solution of system (4.3) coincides with that of the control system (4.6) for all $t \in [t_1 + \tau, +\infty]$. By Lemma 4.1, it follows that $(x_1(t; x_{10}, x_2(\theta)), x_2(t; x_{10}, x_2(\theta))) \to E_2$ as $t \to +\infty$. □

Remarks. Theorem 4.5 states that when $\mathcal{R}_0 > 1$, condition (4.12) holds, and the threshold k is relatively low, the system converges to an endemic equilibrium E_2. Biologically, this implies that if preventive measures are triggered at a lower threshold of infection, the disease still persists but stabilizes at a lower endemic level compared to E_1. This reflects the effect of earlier interventions in reducing the long-term prevalence of the infection in the population. In summary, Theorems 4.4 and 4.5 show how the threshold level k for implementing control measures affects the long-term disease dynamics: higher thresholds lead to a higher endemic prevalence (E_1), while lower thresholds lead to a lower endemic prevalence (E_2).

5. Numerical simulations

In this section, we present numerical simulations to illustrate the main theoretical results of Section 4 and to discuss their biological implications for system (4.3)–(4.4). In addition, we demonstrate that system (1.1) may exhibit certain global bifurcations.

5.1. Numerical results and biological implications

For the parameter values $\mu = 0.2, \beta = 1, \alpha = 1, \gamma = 0.1$ and $k = 0.4$, there exists a pseudo-equilibrium $E_s = (0.4, 0.4) \in \Sigma_s = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid 0.3 < x_1 < 0.448, x_2 = 0.4\}$ for the classical Filippov system (4.3) ($\tau = 0$), where $\mathcal{R}_0 \approx 3.333 > 1$. According to Theorem 2 in [23], E_s is globally asymptotically stable, as illustrated in Fig. 4(a). When $\tau = 0.1 < T_0 = +\infty$, the oscillating region is obtained as $\Sigma_o = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid 0.295 < x_1 < 0.451, x_2 = 0.4\}$, and the corresponding oscillating space M forms a curved quadrilateral bounded by the red and black trajectories in Fig. 4(b). As established in Theorem 4.2, system (4.3) admits a unique

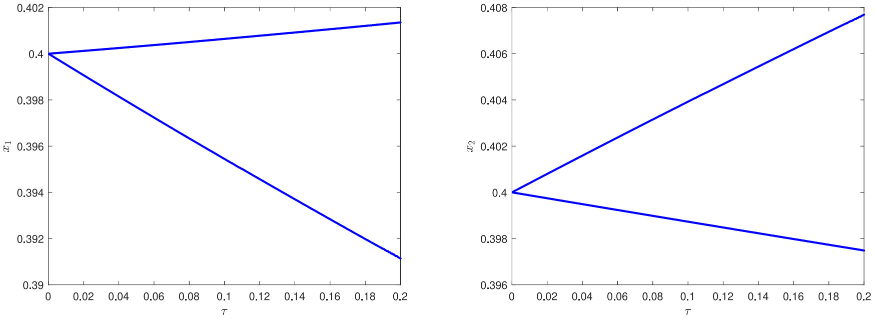


Fig. 5. Bifurcation diagram of system (4.3) with time delay τ as the bifurcation parameter, while all other parameters are as in Fig. 4. The blue lines represent the maximum and minimum values of the limit cycle.

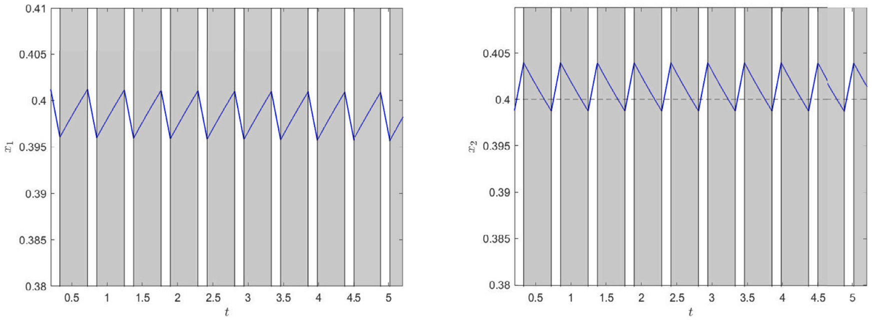
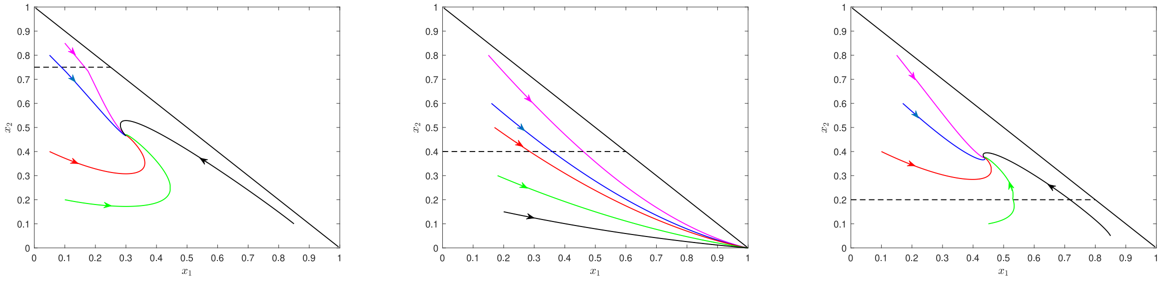


Fig. 6. Time series of $x_1(t)$ and $x_2(t)$ for system (4.3) with $\tau = 0.1$; shaded regions indicate active control periods, and unshaded regions indicate intervals without control. All other parameters are as in Fig. 4.

stable limit cycle within the oscillating space M; this limit cycle is depicted by the green curve in Fig. 4(b). In a neighbourhood of $(x;\tau) = (x_1^*, k; 0) = (0.4, 0.4; 0)$, the amplitude of the limit cycle is asymptotically proportional to τ (see Fig. 5), and its period satisfies $\tilde{T} = 5.456\tau + O(\tau^2)$. The blue lines in Fig. 5 show the maximum and minimum values of the limit cycle. From Fig. 6, we observe that $\tilde{T} \approx 0.5$ when $\tau = 0.1$. The shaded regions in the figure indicate the time intervals during which preventive measures are implemented, while the remaining intervals correspond to the absence of control measures. Biologically, these results suggest that a shorter reporting delay (smaller τ) leads to a smaller limit-cycle amplitude, meaning that timely information dissemination and rapid implementation of preventive measures can significantly mitigate the intensity of epidemic outbreaks.

Choosing $\beta = 0.1$ and keeping all other parameters unchanged, we obtain $\mathcal{R}_0 \approx 0.333 < 1$. According to Theorem 4.3, the disease-free equilibrium E_0 is globally asymptotically stable; hence, the disease will eventually be eradicated regardless of the initial conditions (shown in Fig. 7(a)). When $\beta = 1, k = 0.75 > \bar{k} = 0.7$ or $\beta = 1, k = 0.2 < \tilde{k} \approx 0.291$, while keeping the other parameters unchanged, it follows from Theorems 4.4 and 4.5 that the endemic equilibrium E_1 or E_2 is globally asymptotically stable, respectively (shown in Fig. 7(a) and (c)). This implies that when the threshold level k is either too high or too low, the system converges to a fixed equilibrium and the time delay τ does not influence the qualitative dynamics. To further investigate how the threshold level k affects disease transmission, we take k as the bifurcation parameter and plot the variation of x_1 and x_2 with respect to k (shown in Fig. 8). In this figure, the blue curves represent the maximum and minimum values of the limit cycle. For $\tau = 0.1$, the solution converges to an equilibrium of the free or control subsystem when $k \in (0, x_2^c)$ or $k \in (x_2^f, 1)$, respectively, while a family of stable limit cycles emerges for $k \in [x_2^c, x_2^f]$. These results indicate that the threshold level triggering or suspending control measures plays a crucial role in generating oscillatory epidemic dynamics.

Thus far, we have conducted a theoretical analysis of the local bifurcation behaviour of system (1.1) and verified the results numerically using system (4.3)–(4.4). Our analysis demonstrates the existence of a novel periodic solution induced by the delayed threshold policy. In particular, we have shown that, under appropriate sufficient conditions, the epidemic system stabilizes either at the equilibria of the two subsystems or at the newly generated periodic orbit. It is worth noting that the individual subsystems of (4.3)–(4.4) exhibit relatively simple dynamics: each possesses a globally asymptotically stable endemic equilibrium. In cases where a subsystem admits periodic solutions, we will further illustrate the resulting dynamics in the next subsection through another example.



(a) $\beta = 1, k = 0.75$ (b) $\beta = 0.1, k = 0.4$ (c) $\beta = 1, k = 0.2$

Fig. 7. The limiting behaviour of system (4.3) converges to one of three equilibria: (a) the endemic equilibrium E_1 of the free subsystem (4.5), (b) the disease-free equilibrium E_0, or (c) the endemic equilibrium E_2 of the control subsystem (4.6). All other parameters, except β and k, are identical to those used in Fig. 6.

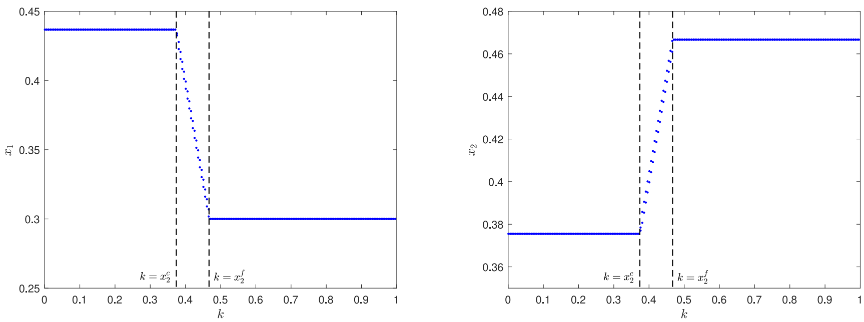


Fig. 8. Bifurcation diagram of system (4.3) with the threshold level k as the bifurcation parameter, while all other parameters are as in Fig. 6. The blue lines represent stable equilibria, while the blue dots represent the maximum and minimum values of the limit cycle.

5.2. Numerical results of global bifurcations

To investigate the global bifurcations of system (1.1), we consider the following specific epidemiological model:

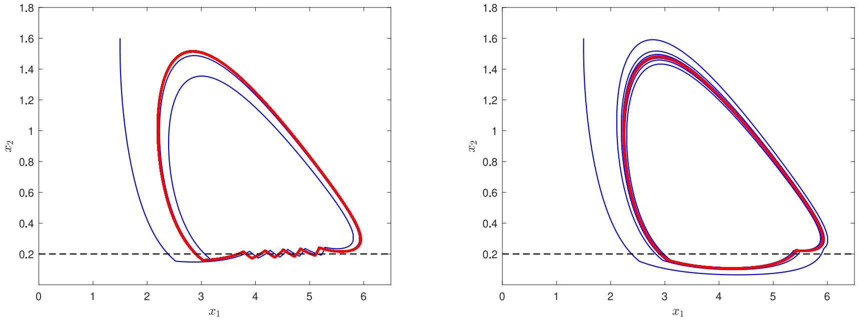
$$\begin{aligned} \frac{dx_1}{dt} &= A - \beta x_1 x_2 - \mu x_1, \\ \frac{dx_2}{dt} &= \beta x_1 x_2 - \upsilon x_2 - \frac{c_1 x_2}{1 + b_1 x_2 + \sigma b_2 c_2}, \end{aligned} \tag{5.1}$$

with

$$\sigma = \begin{cases} 1, & x_2(t-\tau) < k, \\ 0, & x_2(t-\tau) > k, \end{cases}$$

and τ denotes the time delay in the switching decisions. Here, $x_1(t)$ and $x_2(t)$ represent the numbers of susceptible individuals and SARS-infected patients at time t, respectively. The definitions of the remaining parameters are provided in [44]. For this system, one can similarly define the oscillating space M and the oscillating region Σ_o, and it can be shown that a new periodic solution induced by the delayed threshold policy exists; the detailed proof is omitted here for brevity. Motivated by the global bifurcation phenomena observed in the classical planar Filippov system ($\tau = 0$), we now examine the corresponding global bifurcations that occur in the planar Filippov system with time delay in the switching manifold.

It is known that system (5.1) may exhibit a standard periodic solution entirely contained in either G_1 or G_2 through a Hopf bifurcation [44]. As the bifurcation parameter varies, this periodic solution can undergo a topological transformation, leading to the emergence of two distinct types of periodic orbits: periodic solutions that includes a segment within the oscillating space M (oscillating bifurcation), and those that encloses the oscillating region Σ_o (crossing bifurcation). As illustrated in Fig. 9, when $c_2 = 0.8$, a periodic orbit (shown by the red curve) arises via an oscillating bifurcation, characterized by a portion of the trajectory chattering along the switching manifold (Fig. 9(a)). In contrast, when $c_2 = 0.3$, system (5.1) gives rise to a periodic orbit (red curve) through a crossing bifurcation, characterized by the oscillating region surrounding the orbit (Fig. 9(b)). It is noteworthy that these two novel types of periodic solutions — oscillating and crossing bifurcations — are, to the best of our knowledge, the first of their kind ever observed in a planar Filippov system with time delay in the switching manifold.



(a) $c_2 = 0.8$ (b) $c_2 = 0.3$

Fig. 9. Two novel periodic solutions of system (5.1) induced by global bifurcations. Parameters are $A = 2, \beta = 0.8, \mu = 0.1, \upsilon = 0.5, c_1 = 5, b_1 = 1.2, b_2 = 2, k = 0.2$ and $\tau = 0.1$.

6. Discussion

We have investigated a general class of planar Filippov systems with delayed threshold policy, which incorporate the time delay in switching decisions. Compared to the classical Filippov system without delay, this framework provides a more natural and realistic representation of response delays. We first established that system (1.1) with initial condition $x_0 \in \Omega$ is well-posed. Under a relatively small time delay, we demonstrated that system (1.1) can exhibit a slowly oscillating periodic solution under certain conditions by analysing a suitable Poincaré map and rigorously defining the oscillating space. Notably, this newly induced periodic solution and oscillating space correspond, respectively, to the pseudo-equilibrium and sliding region of the classical Filippov system.

We then applied these theoretical results to a specific infectious disease model (4.3)–(4.4), providing methods for identifying the relatively small time delay. We found that a smaller time delay corresponds to a lower amplitude of the periodic solution, suggesting that earlier intervention can mitigate the outbreak. Further, we analysed the global dynamics of the epidemic model with delayed threshold policy and showed that when the threshold level k is either too high or too low, the system converges to an equilibrium in the free or control subsystem, indicating that the time delay τ does not influence the long-term dynamics. Numerical simulations additionally revealed the potential existence of two novel periodic solutions, offering insight for future theoretical studies.

Importantly, unlike the classical planar Filippov system [12,13], system (1.1) with delayed threshold policy has no sliding motion or pseudo-equilibrium. The classical sliding region Σ_s is generalized to an oscillating region Σ_o, which further generates the oscillating space M. Moreover, we established connections between system (1.1) and the classical planar Filippov system: with a relatively small time delay, system (1.1) possesses a unique stable (unstable) limit cycle in M when $\rho < 0$ ($\rho > 0$), corresponding to the uniqueness and stability of the pseudo-equilibrium E_s in the classical Filippov system. Compared with previous epidemic models [11,29], Theorem 4.2 clearly delineates the differences and connections between system (4.3) with and without time delay ($\tau = 0$).

Our work has several limitations, which should be acknowledged. Biologically, delays are not always constant, and switching does not necessarily occur uniformly. Our SIR model assumes equal birth and death rates, which is not always realistic, and relies on mass-action infection, which is appropriate only for well-mixed populations.

This study primarily presents a theoretical analysis and application of planar Filippov systems with delayed threshold policy. For both the general planar Filippov systems (1.1) and the SIR epidemic model (4.3), our analysis mainly focused on slowly oscillating solutions that arise from initial conditions $x_0 \in \bar{\Omega}$. If rapidly oscillating solutions exist, the system may exhibit more complex dynamical behaviour. Here, a rapidly oscillating solution refers to a trajectory for which the time interval between two consecutive switching points is less than τ [45]. Moreover, we explored the conditions under which periodic solutions emerge from global bifurcations, as well as their uniqueness and stability (see Fig. 9).

CRediT authorship contribution statement

Haifeng Wang: Writing – original draft, Investigation, Formal analysis. **Yanni Xiao:** Writing – review & editing, Supervision, Project administration, Methodology, Conceptualization. **Changcheng Xiang:** Software, Methodology, Investigation. **Stacey R. Smith?:** Writing – review & editing, Validation, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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