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CONTROLLING MALARIA WITH INDOOR RESIDUAL SPRAYING IN SPATIALLY HETEROGENEOUS ENVIRONMENTS

Mo'tassem Al-arydah

Department of Mathematics
The University of Ottawa
585 King Edward Ave
Ottawa, ON K1N 6N5, Canada

Robert Smith?

Department of Mathematics and Faculty of Medicine
The University of Ottawa
585 King Edward Ave
Ottawa, ON K1N 6N5, Canada

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Abstract. Indoor residual spraying – spraying insecticide inside houses to kill mosquitoes – has been one of the most effective methods of disease control ever devised, being responsible for the near-eradication of malaria from the world in the third quarter of the twentieth century and saving tens of millions of lives. However, with malaria resurgence currently underway, it has received relatively little attention, been applied only in select physical locations and not always at regular intervals. We extend a time-dependent model of malaria spraying to include spatial heterogeneity and address the following research questions: 1. What are the effects of spraying in different geographical areas? 2. How do the results depend upon the regularity of spraying? 3. Can we alter our control strategies to account for asymmetric phenomena such as wind? We use impulsive partial differential equation models to derive thresholds for malaria control when spraying occurs uniformly, within an interior disc or under asymmetric advection effects. Spatial heterogeneity results in an increase in the necessary frequency of spraying, but control is still achievable.

1. Introduction.

Malaria is an infectious disease that causes morbidity and mortality in the developing world. There are an estimated 360 million cases [29], killing between one to two million people annually [6], primarily among children less than five years of age in sub-Saharan Africa [16]. Three billion people – almost half the world's population – are at risk of malaria [19, 25, 29]. It has been estimated that one in two humans who ever lived has been killed by malaria [10].

In the absence of effective strategies [26], the number of malaria cases might double over the next 20 years. Symptoms include acute febrile illness, chronic debilitation, complication of pregnancy, and impairment of the physical development and learning ability of children. Malaria has an enormous negative social impact

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in high-burden areas [16] and devastating economic effects; in sub-Saharan Africa, malaria costs more than one percentage point of economic growth every year [16].

Malaria is strongly associated with location, with disease transmission restricted to a few kilometres from specific mosquito breeding sites [7]. The clustering of malaria risk has been recognised as a potent factor underlying the robustness of malaria transmission [15]. Conversely, knowledge of locations and individuals at high risk allows specific targeting of intervention measures [13].

Prevention of malaria relies on two main methods: insecticide-treated nets and indoor residual spraying (IRS); both are known to be highly effective [21]. The latter involves spraying houses and structures with insecticides, thereby killing mosquitoes after they have fed, in an effort to stop transmission of the disease. Using these methods, malaria was eradicated or greatly reduced in many countries in the world between the 1940s and 1960s. Due to its success, DDT was rapidly introduced into public health and malaria-control campaigns, and was the main insecticide used in the malaria-eradication campaign carried out between 1955 and 1969 [30]. In sub-Saharan Africa, early malarial eradication pilot projects also showed that malaria was highly responsive to IRS [20].

In recent years, however, IRS has received relatively little attention, despite evidence of its effectiveness in malaria control in countries where it was implemented [14]. Careful delineation of spray areas and populations is necessary for determining the scale of expected impact for each intervention [31]. IRS cannot be used in areas devoid of structures, such as forests or swamps [14]. The application of IRS consistently over time in large areas has altered the vector distribution and subsequently the epidemiological pattern of malaria in Botswana, Namibia, South Africa, Swaziland and Zimbabwe [5, 8, 12, 17]. IRS has commonly been the intervention of choice in areas of a particular economic interest (e.g. tourism, mining, oil extraction, agricultural schemes) that requires a rapid and very effective prevention, where financial and logistic constraints do not prevail [14]. It follows that the spatial heterogeneity of landscape, urban/rural population densities and the distribution of structures plays an important role in the control of malaria.

In this paper, we developed a PDE model to study the effect of IRS, using impulsive partial differential equations, in order to determine the minimal effective spraying period and the amount by which mosquitoes should be reduced at each spraying event. This work is a generalisation of [27], extending our time-dependent spraying model to incorporate spatial effects. We address the following research questions: 1. What are the effects of spraying in different geographical areas? 2. How do the results depend upon the regularity of spraying? 3. Can we alter our control strategies to account for asymmetric phenomena such as wind? To the best of our knowledge, this is the first PDE model of malaria.

This paper is organised as follows. In Section 2, we introduce the problem as a system of PDEs with initial and boundary conditions. In Section 3, we determine steady-state and time-dependent solutions to the PDEs and incorporate impulsive effects. In Section 4, we analyse the impulsive system and examine the effects of fixed and nonfixed spraying, both in our entire region and also within a smaller disc. In Section 5, we find the solution when the wind impact is taken into account. In Section 6, we support our results with numerical simulations. We conclude with a discussion.

2. **The model.** Assume that mosquitoes are either susceptible (M) or infected (N). Their birth rate is Λ and their death rate μ_q, which do not vary significantly if they are infected. The rate of infecting a mosquito is β_m and the number of infected humans is I. Assume that spraying reduces both susceptible and infected mosquitoes by the same portion r, satisfying $0 \le r \le 1$, and that it occurs at distinct times t_k ($k = 0, 1, 2, \ldots$). These times may be fixed or variable. We assume that both susceptible and infected mosquitoes diffuse with the same diffusion constant D. Humans may be susceptible (S), infected (I), or temporarily immune (R). Individuals who have experienced infection may recover without substantial gain in immunity at recovery rate h or may become temporarily immune at acquired immunity rate α. Temporarily immune individuals will become susceptible again at rate δ. The rate of infection of a susceptible individual is β_h. The birth rate for humans is π, the natural death rate is μ_h and the disease-specific death rate is γ. See [27] for further details.

We thus have a system of impulsive PDEs. That is, between impulses t_k, the continuous system of PDEs in the representative area $B(0, \rho_0) \subset \mathbb{R}^2$ (the disc with radius ρ_0 and center at the origin $(0, 0)$) is the following:

$$\begin{aligned}
S_t &= \pi - \beta_h SN + hI + \delta R - \mu_h S && (1)\\
I_t &= \beta_h SN - hI - \alpha I - (\mu_h + \gamma)I && (2)\\
R_t &= \alpha I - \delta R - \mu_h R && (3)\\
M_t &= \Lambda - \mu_q M - \beta_m MI + D\Delta M \qquad t \neq t_k && (4)\\
N_t &= -\mu_q N + \beta_m MI + D\Delta N \qquad t \neq t_k, && (5)
\end{aligned}$$

with the boundary conditions

$$M_\rho(t, \rho_0) = N_\rho(t, \rho_0) = 0 \quad \text{on } \partial B(0, \rho_0), \tag{6}$$

which means that mosquitoes do not enter or leave the disc. Here $\partial B(0, \rho_0)$ represents the boundary of the disc $B(0, \rho_0)$. We refer the interested reader to [2, 3, 4, 18] for more details on the theory of impulsive differential equations.

For $t = t_k$ (impulsive conditions), we have the following:

$$\begin{aligned}
M^+ &= (1 - r)M^- \qquad t = t_k && (7)\\
N^+ &= (1 - r)N^- \qquad t = t_k. && (8)
\end{aligned}$$

Here $()^+$ and $()^-$ are the left and right limits at t_k.

Remark 1. Note that, because of the term $\beta_h SN$ in equation (2), if the number of infected mosquitoes (N) decreases, then the number of infected humans will also decrease. Furthermore, since spraying does not differentiate between susceptible and infected mosquitoes, we will consider the total mosquito population. Thus, our analysis will focus on reducing the total number of mosquitoes as a way of controlling malaria.

Define $\Psi \equiv M + N$ (representing the total number of mosquitoes). Then Ψ satisfies

$$\Psi_t = \Lambda - \mu_q \Psi + D\Delta\Psi \quad \text{in } B(0, \rho_0) \tag{9}$$

with the boundary condition

$$\Psi_\rho(t, \rho_0) = 0 \quad \text{on } \partial B(0, \rho_0) \tag{10}$$

and the impulsive condition

$$\Psi^+ = (1 - r)\Psi^-. \tag{11}$$

To find the exact solution for (9)-(11), we will use classical methods for solving PDEs [1]. We will begin with a particular solution (radially symmetric), then we will look for a general solution.

3. Radially symmetric solution.

Assume that the solution of (9)-(11) is radially symmetric, which means that the solution of this problem in polar coordinates is independent of θ. Thus, $\Psi(\theta,\rho)=\Psi(\rho)$, so $\Delta\Psi=\Psi''+\frac{1}{\rho}\Psi'$, where $\Psi'\equiv\Psi_\rho$.

We will look at the solution as a sum of two solutions, the steady state solution Ψ_s and a time-dependent solution Ψ_d. Therefore, the solution for (9)-(11) can be written as

$$\Psi=\Psi_s+\Psi_d. \tag{12}$$

3.1. Steady-state solution.

Let $t\to\infty$ in (9) and (10), then define Ψ_s as a solution for the resulting steady-state boundary value problem. Then Ψ_s satisfies

$$\begin{aligned}\Lambda-\mu_q\Psi_s+D(\Psi_s''+\frac{1}{\rho}\Psi_s')&=0\\ \Psi_s'(\rho_0)&=0.\end{aligned}$$

This ODE can be written as

$$\rho^2(\Psi_s)''+\rho(\Psi_s)'-\frac{\mu_q}{D}\rho^2\Psi=\frac{-\Lambda}{D}\rho^2. \tag{13}$$

We will look at Ψ_s as a sum of two solutions; i.e.

$$\Psi_s=\Psi_h+\Psi_p, \tag{14}$$

homogeneous and particular solutions, respectively. The homogeneous solution satisfies

$$\rho^2(\Psi_h)''+\rho(\Psi_h)'-\frac{\mu_q}{D}\rho^2\Psi_h=0 \tag{15}$$

and the particular solution is a solution for (13).

By Theorem 1, Appendix A, [1], one solution of (15) is of the form

$$\Psi_h(\rho)=\sum_{n=0}^{\infty}a_n\rho^{s+n}, \tag{16}$$

with $a_0\neq 0$ (arbitrary constant), $s\in\mathbb{R}$.

Since (16) satisfies (15), we have

$$\sum_{n=0}^{\infty}a_n(s+n)(s+n-1)\rho^{s+n}+\sum_{n=0}^{\infty}a_n(s+n)\rho^{s+n}-\frac{\mu_q}{D}\sum_{n=0}^{\infty}a_n\rho^{s+n+2}=0,$$

which is equivalent to

$$a_0(s(s-1)+s)\rho^s+a_1((s+1)s+(s+1))\rho^{s+1}+\sum_{n=2}^{\infty}(-\frac{\mu_q}{D}a_{n-2}+(s+n)^2a_n)\rho^{s+n}=0. \tag{17}$$

It is clear that all terms in (17) are zeros so, from the coefficient of ρ^s, we have $s=0$. The coefficient of ρ^{s+1} implies $a_1=0$. Finally, the last term implies $a_n=\frac{\mu_q}{D}\frac{a_{n-2}}{n^2}$ for all $n\geq 2$. In other words, for all $n\geq 0$, we have $a_{2n+1}=0$ and $a_{2n}=\frac{(\frac{\mu_q}{D})^n}{(2^n n!)^2}a_0$. Therefore, we have

$$\Psi_h(\rho)=a_0\sum_{n=0}^{\infty}\frac{(\frac{\mu_q}{D})^n\rho^{2n}}{(2^n n!)^2}. \tag{18}$$

Another independent solution for (15) is $\tilde{\Psi}_h = \Psi_h \times \ln(\rho) + \sum_{m=1}^{\infty} b_m \rho^m$ (see Theorem 2, Appendix A, [1]), which is ignored here because it is singular at the origin.

Note that any particular solution for (13) has the form

$$\Psi_p(\rho) = A\rho^2 + B\rho + C, \tag{19}$$

for some constants A, B and C. Since (19) satisfies (15), we have

$$-\frac{\mu_q}{D}A\rho^4 - \frac{\mu_q}{D}B\rho^3 + (4A - \frac{\mu_q}{D}C)\rho^2 + B\rho = -\frac{\Lambda}{D}\rho^2,$$

which implies $A = B = 0$ and $C = \frac{\Lambda}{\mu_q}$. Therefore,

$$\Psi_p = \frac{\Lambda}{\mu_q}. \tag{20}$$

As a result of (14), (18) and (20) imply

$$\Psi_s = \frac{\Lambda}{\mu_q} + c_1 \sum_{n=0}^{\infty} \frac{(\frac{\mu_q}{D})^n \rho^{2n}}{(2^n n!)^2}.$$

Now, using the boundary condition $\Psi_s'(\rho_0) = 0$, we conclude that $c_1 = 0$. Therefore,

$$\Psi_s = \frac{\Lambda}{\mu_q}. \tag{21}$$

Remark 2. Note that if we analyse the system without impulses, then the same idea above can be used to prove that the disease-free equilibrium ($I = 0$) for the nonimpulsive model (4), (5) and (6) is given by $E_0 = (\overline{M}, \overline{N}) = \left(\frac{\Lambda}{\mu_q}, 0\right)$. This result agrees with what we have in Equation 3.1, [27].

3.2. **Time-dependent solution.** The time-dependent solution Ψ_d satisfies

$$(\Psi_d)_t = -\mu_q \Psi_d + D(\Psi_d'' + \frac{1}{\rho}\Psi_d'), \tag{22}$$

and the two conditions (10) and (11).

To calculate the solution of this problem, we will use the separation of variables method, so let $\Psi_d(t,\rho) = T(t)R(\rho)$. If we substitute this in (22), then we have

$$\frac{T'}{DT} + \frac{\mu_q}{D} = \frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R}. \tag{23}$$

Since the left-hand side in (23) depends on t only and the right-hand side depends on ρ only, then we must have

$$\frac{T'}{DT} + \frac{\mu_q}{D} = k \tag{24}$$

and

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} = k, \tag{25}$$

for some constant k. Also, the boundary condition (10) implies

$$R'(\rho_0) = 0. \tag{26}$$

Now consider the two cases:

Case 1. $k = 0$.

In this case, (24) implies $T' + \mu_q T = 0$. Therefore, $T(t) = T_0 \exp(-\mu_q t)$, for some constant T_0. Also, (25) becomes $(\rho R')' = 0$, which implies $\rho R' = C$, for some constant C. Moreover, from (26), we have $C = 0$. Therefore, we have $R(\rho) = \rho^*$, for

some constant ρ^*. As a result of this case, we have $\Psi(t,\rho) = \Psi(t) = c_0 \exp(-\mu_q t)$, for some constant c_0.
Case 2. $k \neq 0$.
In this case, (24) can be written as $T' + (\mu_q - kD)T = 0$, which has the solution $T(t) = T_1 \exp((kD - \mu_q)t)$, for some constant T_1. Note that the solution Ψ should satisfy, $\Psi \to \Psi_s$ (unconditionally) as $t \to \infty$, so k must be of the form $k = -\lambda^2$, for some constant λ. Also, equation (25) can be written in the form

$$\rho^2 R'' + \rho R' + \lambda^2 \rho^2 R = 0. \tag{27}$$

To find $R(\rho)$ that satisfies (26) and (27), we try a solution of the form (16) and conclude that

$$R(\rho) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda\rho)^{2n}}{(2^n n!)^2} \equiv a_0 J_0(\lambda\rho),$$

which is a Bessel function of the first kind [1]. Again we ignored the second independent singular solution; that is, the solution of the form $\tilde{R}(\rho) = R(\rho)\ln(\rho) + \sum_{n=1}^{\infty} b_n \rho^n$. Therefore, the set of all regular solutions for (27) is

$$R(\rho) = c_2 J_0(\lambda\rho).$$

But $R'(\rho_0) = 0$, so we have $c_2 \lambda J_0'(\lambda\rho_0) = -c_2\lambda J_1(\lambda\rho_0) = 0$, so $c_2 \neq 0$ only when $\lambda = \lambda_n = \frac{z_n'}{\rho_0}$, where z_n' $(n = 1, 2, 3, \ldots)$ is an increasing sequence of positive roots of $J_0'(\rho) = -J_1(\rho)$ (the second Bessel function of the first kind). As a result of the two cases, we have

$$\Psi_d(t,\rho) = c_0 \exp(-\mu_q t) + \sum_{n=1}^{\infty} c_n \exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)t) J_0(\frac{z_n'}{\rho_0}\rho), \tag{28}$$

3.3. General solution and impulsive condition.

From (12), (21) and (28), the general solution for (9) and (10) (the boundary value problem between impulses) is

$$\Psi(t,\rho) = \frac{\Lambda}{\mu_q} + c_0 \exp(-\mu_q t) + \sum_{n=1}^{\infty} c_n \exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)t) J_0(\frac{z_n'}{\rho_0}\rho). \tag{29}$$

Next, we will use the impulsive conditions (11) to determine the value of c_n for $n = 0, 1, 2, \ldots$ Note that c_n will depend also on k; for this reason, we will replace c_n by $c_{n,k}$.

For $n = 0$, we will use the fact that $J_0(0) = 1$, so (29) and the impulsive conditions imply

$$c_{0,k} = (\Psi(t_k^+, 0) - \frac{\Lambda}{\mu_q}) \exp(\mu_q t_k) - \sum_{n=1}^{\infty} c_{n,k} \exp(-\left|\frac{z_n'}{\rho_0}\right|^2 D t_k), \tag{30}$$

which means that $c_{0,k}$ depends on $c_{n,k}$ for $n \geq 1$.

For $n \geq 1$, we will use the fact that

$$\int_0^{\rho_0} \rho J_1(\frac{z_n'}{\rho_0}\rho) J_1(\frac{z_m'}{\rho_0}\rho) d\rho = \frac{1}{2}\rho_0^2 J_2^2(z_n')\delta_{n,m}, \tag{31}$$

which can be found in [9], page 285.

From (29) and the impulsive conditions, we have

$$\Psi(t_k^+, \rho) - \frac{\Lambda}{\mu_q} - c_{0,k}\exp(-\mu_q t_k) = \sum_{n=1}^{\infty} c_{n,k} \exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)t_k) J_0(\frac{z_n'}{\rho_0}\rho). \tag{32}$$

Now differentiate both sides of (32) with respect to ρ. Then we have

$$\Psi_\rho(t_k^+,\rho) = -\sum_{n=1}^{\infty} \frac{z'_n}{\rho_0} c_{n,k} \exp((-\mu_q - \left|\frac{z'_n}{\rho_0}\right|^2 D)t_k) J_1(\frac{z'_n}{\rho_0}\rho). \tag{33}$$

Multiplying (33) by $\rho J_1(\frac{z'_n}{\rho_0}\rho)$ and integrating over $[0,\rho]$, then using (31) with some rearranging, we conclude

$$c_{n,k} = \frac{-2}{\rho_0 z'_n J_2^2(z'_n)} \exp((\mu_q + \left|\frac{z'_n}{\rho_0}\right|^2 D)t_k) \int_0^{\rho_0} \rho \Psi_\rho(t_k^+,\rho) J_1(\frac{z'_n}{\rho_0}\rho) d\rho. \tag{34}$$

Then (29), (30) and (34) imply

$$\begin{aligned}\Psi(t,\rho) = {} & \frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t-t_k))] \\ & + [\Psi(t_k^+,0) + \sum_{n=1}^{\infty} \frac{2}{\rho_0 z_n J_2^2(z'_n)} \int_0^{\rho_0} \rho \Psi_\rho(t_k^+,\rho) J_1(\frac{z'_n}{\rho_0}\rho) d\rho] \exp(-\mu_q(t-t_k)) \\ & - \sum_{n=1}^{\infty} \frac{2}{\rho_0 z'_n J_2^2(z'_n)} \exp((-\mu_q - \left|\frac{z'_n}{\rho_0}\right|^2 D)(t-t_k)) \int_0^{\rho_0} \rho \Psi_\rho(t_k^+,\rho) J_1(\frac{z'_n}{\rho_0}\rho) d\rho.\end{aligned}$$

The impulsive conditions (11) can be written as

$$\Psi(t_k^+,\rho) = (1-r)\Psi(t_k^-,\rho), \tag{35}$$

where

$$\Psi(t_0^+,\rho) \equiv (1-r)\Psi_0(\rho). \tag{36}$$

As a result, we have

$$\begin{aligned}\Psi(t_{k+1}^-,\rho) &= \frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_{k+1}-t_k))] \\ &\quad + [\Psi(t_k^+,0) - \sum_{n=1}^{\infty} b_{n,k}] \exp(-\mu_q(t_{k+1}-t_k)) \\ &\quad + \sum_{n=1}^{\infty} b_{n,k} \exp((-\mu_q - \left|\frac{z'_n}{\rho_0}\right|^2 D)(t_{k+1}-t_k)). \\ &= \frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_{k+1}-t_k))] \\ &\quad + [(1-r)\Psi(t_k^-,0) - \sum_{n=1}^{\infty} b_{n,k}] \exp(-\mu_q(t_{k+1}-t_k)) \\ &\quad + \sum_{n=1}^{\infty} b_{n,k} \exp((-\mu_q - \left|\frac{z'_n}{\rho_0}\right|^2 D)(t_{k+1}-t_k)),\end{aligned} \tag{37}$$

where

$$\begin{aligned}b_{n,k} &= \frac{-2}{\rho_0 z'_n J_2^2(z'_n)} \int_0^{\rho_0} \rho \Psi_\rho(t_k^+,\rho) J_1(\frac{z'_n}{\rho_0}\rho) d\rho \\ &= \frac{-2}{\rho_0 z'_n J_2^2(z'_n)} (1-r) \int_0^{\rho_0} \rho \Psi_\rho(t_k^-,\rho) J_1(\frac{z'_n}{\rho_0}\rho) d\rho.\end{aligned} \tag{38}$$

As a result, we have a recurrence relation for the total number of mosquitoes at any point with distance ρ from the origin, immediately before spraying (equations (37) and (38)), under the assumption that the solution is radially symmetric. This relation depends on the birth and death rates of mosquitoes, the spraying times, and the spraying effectiveness.

Note also that, since spraying always reduces the total number of mosquitoes, the solution immediately before spraying will be a local maximum.

Remark 3. Note that if $\Psi_0(\rho) = C$ is constant in ρ, then $b_{n,k} = 0$ for all $n \geq 1$, because $(\Psi_0)_\rho = 0$. Therefore, (37) becomes

$$\begin{aligned}\Psi(t_{k+1}^-, \rho) = \frac{\Lambda}{\mu_q}&[1 - \exp(-\mu_q(t_{k+1} - t_k))]\\ &+ (1 - r)\Psi(t_k^-, 0)\exp(-\mu_q(t_{k+1} - t_k)),\end{aligned}$$

which is similar to what we have in the nonspatial model [27].

In the following example, we introduce a simple function Ψ_0 to give the reader an idea about evaluating the coefficients $b_{n,k}$ given by the formula (38).

Example. Let $\Psi_0(\rho) = \rho_0^2 - \rho^2 + \alpha$ (ρ_0 and α are positive constants), which means we have more mosquitoes in the center of the circle, under the assumption that we have more houses and farms at the center and mosquitoes prefer to live in these places. This distribution is illustrated in Figure 1.

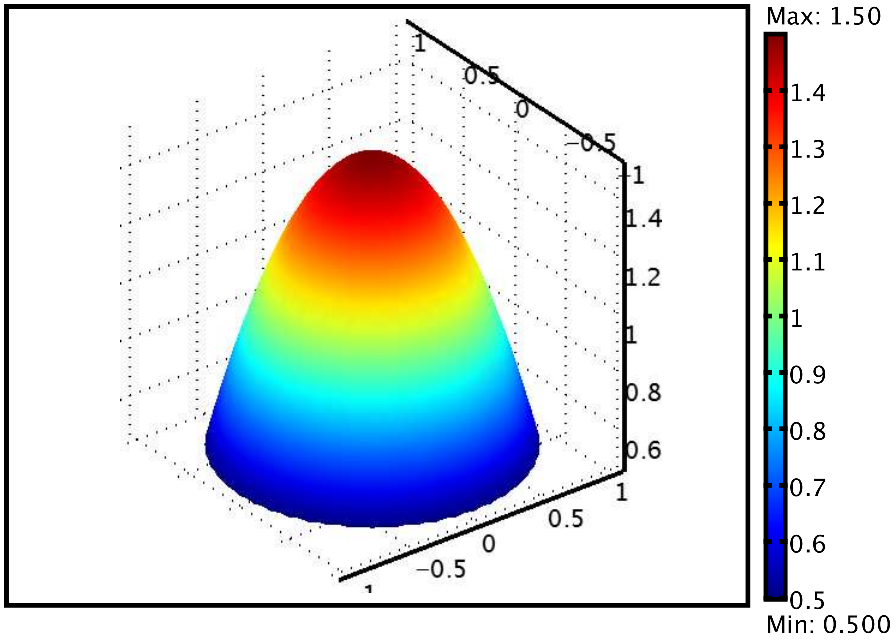

FIGURE 1. A centre-focused distribution of mosquitoes. Here, $\Psi_0(\rho)$ for $\rho_0 = 1$, $\alpha = 0.5$ and $0 \leq \rho \leq 1$.

Then we have

$$\begin{aligned} b_{n,0} &= \frac{4}{\rho_0 z_n' J_2^2(z_n')}(1-r)\int_0^{\rho_0} \rho^2 J_1(\frac{z_n'}{\rho_0}\rho)d\rho \\ &= \frac{4\rho_0^2}{(z_n')^4 J_2^2(z_n')}(1-r)\int_0^{z_n'} u^2 J_1(u)du \\ &= \frac{4\rho_0^2}{(z_n')^4 J_2^2(z_n')}(1-r)\left[u^2 J_2^2(u)\right]_0^{z_n'} \\ &= \frac{4\rho_0^2}{(z_n')^2}(1-r). \end{aligned}$$

Here, we used integration by substitution and the fact that $\frac{d}{d\rho}(\rho^2 J_2^2(\rho)) = \rho^2 J_1(\rho)$. (See [9], page 282.) Therefore,

$$\begin{aligned} \Psi(t_1^-,\rho) = \frac{\Lambda}{\mu_q}&[1-\exp(-\mu_q(t_1-t_0))] + (1-r)\Psi_0(0)\exp(-\mu_q(t_1-t_0)) \\ &- 4\rho_0^2(1-r)\exp(-\mu_q(t_1-t_0))\sum_{n=1}^{\infty}\frac{1}{(z_n')^2} \\ &+ 4\rho_0^2(1-r)\sum_{n=1}^{\infty}\frac{1}{(z_n')^2}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_1-t_0)). \end{aligned}$$

This function represents the number of mosquitoes immediately before the first spraying.

4. Symmetric patterns of spraying.

We now examine a number of different options for spraying, that vary with time and space, but where the pattern is symmetric. Spatially, we shall look at uniform spraying and then spraying within an interior region. However, we shall also examine time-dependency of spraying. Ideally, spraying would occur at fixed, regular intervals. However, due to infrastructure and resource limitations, it may not be possible to spray periodically. Furthermore, the past history of spraying events may not be available, meaning that we need to derive a solution for nonfixed spraying, under the constraint that only limited, recent information may be available.

4.1. Spraying is applied everywhere and occurs at fixed times.

First we shall examine the case most analagous to [27], where the landscape is homogeneous, so that spraying can be applied uniformly. We shall also examine the idealised case, when spraying can be applied at regular periods.

From (37) and (38), we have

$$\begin{aligned} \Psi_1^- &\equiv \Psi(t_1^-,\rho) \\ &= \frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_1-t_0))] + (1-r)\Psi_0(0)\exp(-\mu_q(t_1-t_0)) \\ &\quad - \exp(-\mu_q(t_1-t_0))\sum_{n=1}^{\infty} b_{n,0} + \sum_{n=1}^{\infty} b_{n,0}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_1-t_0))J_0(\frac{z_n'}{\rho_0}\rho), \end{aligned}$$

where

$$b_{n,0} = \frac{-2}{\rho_0 z_n' J_2^2(z_n')}(1-r)\int_0^{\rho_0} \rho(\Psi_0)_\rho(\rho) J_1(\frac{z_n'}{\rho_0}\rho)d\rho.$$

Thus

$$\begin{aligned}
\Psi_2^- = {} & \frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_2-t_1))] + (1-r)\Psi(t_1^-,0)\exp(-\mu_q(t_2-t_1)) \\
& - \exp(-\mu_q(t_2-t_1))\sum_{n=1}^{\infty} b_{n,1} + \sum_{n=1}^{\infty} b_{n,1}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_2-t_1))J_0(\frac{z_n'}{\rho_0}\rho) \\
= {} & \frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_2-t_1))] + (1-r)\left(\frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_1-t_0))]\right. \\
& + (1-r)\Psi_0(0)\exp(-\mu_q(t_1-t_0)) - \exp(-\mu_q(t_1-t_0))\sum_{n=1}^{\infty} b_{n,0} \\
& \left. + \sum_{n=1}^{\infty} b_{n,0}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_1-t_0))\right)\exp(-\mu_q(t_2-t_1)) \\
& - \exp(-\mu_q(t_2-t_1))\sum_{n=1}^{\infty} b_{n,1} + \sum_{n=1}^{\infty} b_{n,1}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_2-t_1))J_0(\frac{z_n'}{\rho_0}\rho),
\end{aligned}$$

where

$$\begin{aligned}
b_{n,1} = {} & \frac{-2}{\rho_0 z_n' J_2^2(z_n')}(1-r)\int_0^{\rho_0}\rho\left(\sum_{s=1}^{\infty} b_{s,0}\exp((-\mu_q - (\frac{z_s'}{\rho_0})^2 D)(t_1-t_0))\right. \\
& \left. -\frac{z_s'}{\rho_0}J_1(\frac{z_s'}{\rho_0}\rho)\right) J_1(\frac{z_n'}{\rho_0}\rho)d\rho \\
= {} & \frac{-2}{\rho_0 z_n' J_2^2(z_n')}(1-r)b_{n,0}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_1-t_0)) - \frac{z_n'}{\rho_0}\frac{\rho_0^2}{2}J_2^2(z_n') \\
= {} & (1-r)\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_1-t_0))b_{n,0},
\end{aligned}$$

and where we have used (31). Therefore, we have

$$\begin{aligned}
\Psi_2^- = {} & \frac{\Lambda}{\mu_q}[1 - r\exp(-\mu_q(t_2-t_1)) - (1-r)\exp(-\mu_q(t_2-t_0))] \\
& + (1-r)^2\Psi_0(0)\exp(-\mu_q(t_2-t_0)) - (1-r)\exp(-\mu_q(t_2-t_0))\sum_{n=1}^{\infty} b_{n,0} \\
& + (1-r)\exp(-\mu_q(t_2-t_0))\sum_{n=1}^{\infty} b_{n,0}\exp(-\left|\frac{z_n'}{\rho_0}\right|^2 D(t_1-t_0)) \\
& - (1-r)\exp(-\mu_q(t_2-t_0))\sum_{n=1}^{\infty} b_{n,0}\exp(-\left|\frac{z_n'}{\rho_0}\right|^2 D(t_1-t_0)) \\
& + (1-r)\exp(-\mu_q(t_2-t_0))\sum_{n=1}^{\infty} b_{n,0}\exp(-\left|\frac{z_n'}{\rho_0}\right|^2 D(t_2-t_0))J_0(\frac{z_n'}{\rho_0}\rho),
\end{aligned}$$

which implies

$$\begin{aligned}\Psi_2^- &= \frac{\Lambda}{\mu_q}[1 - r\exp(-\mu_q(t_2 - t_1)) - (1-r)\exp(-\mu_q(t_2 - t_0))] \\ &\quad + (1-r)^2\Psi_0(0)\exp(-\mu_q(t_2 - t_0)) - (1-r)\exp(-\mu_q(t_2 - t_0))\sum_{n=1}^{\infty} b_{n,0} \\ &\quad + (1-r)\exp(-\mu_q(t_2 - t_0))\sum_{n=1}^{\infty} b_{n,0}\exp(-\left|\frac{z'_n}{\rho_0}\right|^2 D(t_2 - t_0))J_0(\frac{z'_n}{\rho_0}\rho).\end{aligned}$$

We now repeat the process to get

$$\begin{aligned}\Psi_3^- &= \frac{\Lambda}{\mu_q}[1 - \exp(-\mu_q(t_3 - t_2))] + (1-r)\Psi(t_2^-, 0)\exp(-\mu_q(t_3 - t_2)) \\ &\quad - \exp(-\mu_q(t_3 - t_2))\sum_{n=1}^{\infty} b_{n,2} + \sum_{n=1}^{\infty} b_{n,2}\exp((-\mu_q - \left|\frac{z'_n}{\rho_0}\right|^2 D)(t_3 - t_2))J_0(\frac{z'_n}{\rho_0}\rho) \\ &= \frac{\Lambda}{\mu_q}[1 - r\exp(-\mu_q(t_3 - t_2)) - r(1-r)\exp(-\mu_q(t_3 - t_1)) \\ &\quad - (1-r)^2\exp(-\mu_q(t_3 - t_0))] + (1-r)^3\Psi_0(0)\exp(-\mu_q(t_3 - t_0)) \\ &\quad - (1-r)^2\exp(-\mu_q(t_3 - t_0))\sum_{n=0}^{\infty} b_{n,0} \\ &\quad + (1-r)^2\exp(-\mu_q(t_3 - t_0))\sum_{n=0}^{\infty} b_{n,0}\exp(-\left|\frac{z'_n}{\rho_0}\right|^2 D(t_2 - t_0)) \\ &\quad - \exp(-\mu_q(t_3 - t_2))\sum_{n=1}^{\infty} b_{n,2} + \sum_{n=1}^{\infty} b_{n,2}\exp(-\mu_q - \left|\frac{z'_n}{\rho_0}\right|^2 D(t_3 - t_2))J_0(\frac{z'_n}{\rho_0}\rho),\end{aligned}$$

with

$$\begin{aligned}b_{n,2} &= \frac{-2}{\rho_0 z'_n J_2^2(z'_n)}(1-r)\int_0^{\rho_0} \rho\left((1-r)\exp(-\mu_q(t_2 - t_0))\sum_{s=1}^{\infty} b_{s,0}\right. \\ &\qquad \left.\times\exp(-(\frac{z'_s}{\rho_0})^2 D(t_2 - t_0)) - \frac{z'_s}{\rho_0}J_1(\frac{z'_s}{\rho_0}\rho)\right)J_1(\frac{z'_n}{\rho_0}\rho)d\rho \\ &= (1-r)^2\exp(-\mu_q(t_2 - t_0))\exp(-\left|\frac{z'_n}{\rho_0}\right|^2 D(t_2 - t_0))b_{n,0}.\end{aligned}$$

Then,

$$\begin{aligned}\Psi_3^- &= \frac{\Lambda}{\mu_q}[1 - r\exp(-\mu_q(t_3 - t_2)) - r(1-r)\exp(-\mu_q(t_3 - t_1)) \\ &\quad - (1-r)^2\exp(-\mu_q(t_3 - t_0))] + (1-r)^3\Psi_0(0)\exp(-\mu_q(t_3 - t_0)) \\ &\quad - (1-r)^2\exp(-\mu_q(t_3 - t_0))\sum_{n=0}^{\infty} b_{n,0} \\ &\quad + (1-r)^2\sum_{n=1}^{\infty} b_{n,0}\exp((-\mu_q - \left|\frac{z'_n}{\rho_0}\right|^2 D)(t_3 - t_0))J_0(\frac{z'_n}{\rho_0}\rho).\end{aligned}$$

Similarly,

$$\begin{aligned}\Psi_4^- &= \frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_4-t_3))] + (1-r)\Psi(t_3^-,0)\exp(-\mu_q(t_4-t_3)) \\ &\quad - \exp(-\mu_q(t_4-t_3))\sum_{n=1}^{\infty} b_{n,3} + \sum_{n=1}^{\infty} b_{n,3}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_4-t_3))J_0(\frac{z_n'}{\rho_0}\rho)\end{aligned}$$

where

$$b_{n,3} = (1-r)^3\exp(-\mu_q(t_3-t_0))\exp(-\left|\frac{z_n'}{\rho_0}\right|^2 D(t_3-t_0))b_{n,0}.$$

Hence,

$$\begin{aligned}\Psi_4^- &= \frac{\Lambda}{\mu_q}[1-r\exp(-\mu_q(t_4-t_3)) - r(1-r)\exp(-\mu_q(t_4-t_2)) \\ &\quad - r(1-r)^2\exp(-\mu_q(t_4-t_1)) - (1-r)^3\exp(-\mu_q(t_4-t_0))] \\ &\quad + (1-r)^4\Psi_0(0)\exp(-\mu_q(t_4-t_0)) - (1-r)^3\exp(-\mu_q(t_4-t_0))\sum_{n=0}^{\infty} b_{n,0} \\ &\quad + (1-r)^3\sum_{n=1}^{\infty} b_{n,0}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_4-t_0))J_0(\frac{z_n'}{\rho_0}\rho).\end{aligned}$$

We can thus derive the general solution for the value immediately before the mth spraying:

$$\begin{aligned}\Psi_m^- &= \frac{\Lambda}{\mu_q}[1-\sum_{i=1}^{m-1} r(1-r)^{m-i-1}\exp(-\mu_q(t_m-t_i)) \\ &\quad - (1-r)^{m-1}\exp(-\mu_q(t_m-t_0))] \\ &\quad + (1-r)^m\Psi_0(0)\exp(-\mu_q(t_m-t_0)) - (1-r)^{m-1}\exp(-\mu_q(t_m-t_0))\sum_{n=0}^{\infty} b_{n,0} \\ &\quad + (1-r)^{m-1}\sum_{n=1}^{\infty} b_{n,0}\exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_m-t_0))J_0(\frac{z_n'}{\rho_0}\rho). \qquad (39)\end{aligned}$$

If spraying occurs at fixed times, so that $\tau = t_{s+1} - t_s$, then $t_m - t_i = (m-i)\tau$. Therefore, we have

$$\begin{aligned}\Psi_m^- &= \frac{\Lambda}{\mu_q}[1-\frac{r\exp(-\mu_q\tau) - r(1-r)^{m-1}\exp(-\mu_q m\tau)}{1-(1-r)\exp(-\mu_q\tau)} \\ &\quad - (1-r)^{m-1}\exp(-\mu_q m\tau)] \\ &\quad + (1-r)^m\Psi_0(0)\exp(-\mu_q m\tau) - (1-r)^{m-1}\exp(-\mu_q m\tau)\sum_{n=0}^{\infty} b_{n,0} \\ &\quad + (1-r)^{m-1}\exp(-\mu_q m\tau)\sum_{n=1}^{\infty} b_{n,0}\exp(-\left|\frac{z_n'}{\rho_0}\right|^2 Dm\tau)J_0(\frac{z_n'}{\rho_0}\rho).\end{aligned}$$

Since $0 < r \leq 1$, we have proved the following theorem.

Theorem 4.1. *The total mosquito population satisfies*

$$\lim_{m\to\infty}\Psi_m^- = \frac{\Lambda}{\mu_q}[1-\frac{r\exp(-\mu_q\tau)}{1-(1-r)\exp(-\mu_q\tau)}].$$

Remark 4. Note that $\Psi_m^- \to 0$ as $\tau \to 0$ and $m \to \infty$, which means that the total mosquito population shrinks to zero as the spraying period decreases.

We can reduce the total mosquito population below a desired threshold $\tilde{\Psi}$ per unit area, if the minimum spraying effectiveness satisfies

$$\tilde{r} = 1 - [1 - \frac{\Lambda}{\mu_q \tilde{\Psi}}(1 - \exp(-\mu_q \tau))] \exp(\mu_q \tau),$$

or if the minimum spraying period satisfies

$$\tilde{\tau} = -\frac{1}{\mu_q} \ln[\frac{\Lambda - \mu_q \tilde{\Psi}}{\Lambda + \mu_q \tilde{\Psi}(r-1))}],$$

which are similar to the results given in Corollary 4.3, [27].

Note that the total number of mosquitoes is independent of D, because as $m \to \infty$ we have also $t \to \infty$. The number of mosquitoes is thus in steady state, where no diffusion occurs.

It follows that we can find the minimal spraying effectiveness or the minimal spraying period for fixed spraying in the entire region, in terms of the birth and death rates of mosquitoes and the spraying effectiveness. The results agree with the nonspatial model [27]. Therefore, our work proves that some of the results in [27] apply to the spatial model, when spraying is applied everywhere and a symmetric solution is considered.

4.2. **Spraying is applied everywhere and occurs at nonfixed times.**

Note that, for large m, all terms in (39) containing $(1-r)^{m-1}$ or $(1-r)^m$ are very small because $0 < r \leq 1$. Therefore,

$$\begin{aligned} \Psi_m^- &\approx \frac{\Lambda}{\mu_q}[1 - \sum_{i=1}^{m-1} r(1-r)^{m-i-1} \exp(-\mu_q(t_m - t_i))] \\ &< \frac{\Lambda}{\mu_q}[1 - r(1-r)^{m-(m-1)-1} \exp(-\mu_q(t_m - t_{m-1}))] \\ &= \frac{\Lambda}{\mu_q}[1 - r \exp(-\mu_q(t_m - t_{m-1}))]. \end{aligned} \tag{40}$$

Also

$$\begin{aligned} \Psi_{m+1}^- = &\frac{\Lambda}{\mu_q}[1 - \exp(-\mu_q(t_{m+1} - t_m))] \\ &+ [(1-r)\Psi(t_m^-, 0) - \sum_{n=1}^{\infty} b_{n,m}] \exp(-\mu_q(t_{m+1} - t_m)) \\ &+ \sum_{n=1}^{\infty} b_{n,m} \exp((-\mu_q - \left|\frac{z_n'}{\rho_0}\right|^2 D)(t_{m+1} - t_m)), \end{aligned}$$

where

$$b_{n,m} = (1-r)^m \exp(-\mu_q(t_m - t_0)) \exp(-\left|\frac{z_n'}{\rho_0}\right|^2 D(t_m - t_0)) b_{n,0},$$

which is very small for large m because $0 < r \leq 1$. This implies

$$\Psi_{m+1}^- \approx \frac{\Lambda}{\mu_q}[1 - \exp(-\mu_q(t_{m+1} - t_m))] + (1-r)\Psi(t_m^-, 0) \exp(-\mu_q(t_{m+1} - t_m)). \tag{41}$$

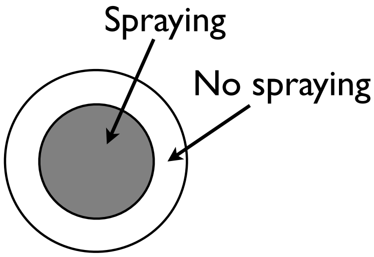

FIGURE 2. Spraying in an interior disc.

From (40) and (41), we have

$$\begin{aligned}\Psi^-_{m+1} &< \frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_{m+1}-t_m))]\\ &\quad + (1-r)\frac{\Lambda}{\mu_q}[1-r\exp(-\mu_q(t_m-t_{m-1}))]\exp(-\mu_q(t_{m+1}-t_m)),\\ &\equiv \tilde{\Psi}.\end{aligned}$$

Solving for t_{m+1}, we have

$$t_{m+1} = t_m - \frac{1}{\mu_q}\ln\left[\frac{2-r-\frac{\tilde{\Psi}\mu_q}{\Lambda}}{1+r(1-r)\exp(-\mu_q(t_m-t_{m-1}))}\right],$$

which gives the time at which the spraying reduces the number of mosquitoes to be less than $\tilde{\Psi}$ (a desired value per unit area). Note that to find such time we need to know the previous two spraying times. This is exactly what we have in Theorem 4.4, [27].

Thus, the "next best" spraying events for nonfixed spraying is derived, by assuming that the time between the current spraying and two sprayings events previously is sufficiently large. Again, we prove that the results in [27] are analogous in the spatial model when a nonfixed spraying is applied everywhere and a symmetric solution is considered.

The following theorem then follows immediately.

Theorem 4.2. *If nonfixed spraying occurs indefinitely, then there exists a minimum spraying effectiveness* r_0*, satisfying* $0 < r_0 < 1$*, such that variable spraying is only effective for* $r_0 \le r < 1$*. Furthermore, on this interval, the minimum spraying interval for indefinite nonfixed spraying is always less than the minimum spraying interval for regular spraying.*

Proof. See Theorem 4.5, [27]. ☐

4.3. **Fixed spraying in an interior disc.** Let $0 < \rho_{00} < \rho_0$ and assume that spraying is applied only in the disc $B(0,\rho_{00})$. See Figure 2. Moreover, the flux on the boundary of the center circle depends on both time and diffusion. Thus, if τ and D are sufficiently small, mosquitoes will not have enough time to diffuse to the disc $B(0,\rho_{00})$ from the outer region $B(0,\rho_0)\setminus B(0,\rho_{00})$; we shall thus assume that this effect is negligible.

In this case, the impulsive conditions (35) and (36) are applied to the disc $B(0,\rho_{00})$ and the initial condition $\Psi(t_0,\rho) = \Psi_0(\rho)$ is applied to the annulus $B(0,\rho_0) \setminus B(0,\rho_{00})$. This implies that the impulsive solution in $B(0,\rho_{00})$ is

$$\begin{aligned}\underline{\Psi}(t_{k+1}^-,\rho) = \frac{\Lambda}{\mu_q}&[1-\exp(-\mu_q(t_{k+1}-t_k))]\\ &+[(1-r)\Psi(t_k^-,0)-\sum_{n=1}^{\infty} b_{n,k}]\exp(-\mu_q(t_{k+1}-t_k))\\ &+\sum_{n=1}^{\infty} b_{n,k}\exp((-\mu_q-\left|\frac{z_n'}{\rho_0}\right|^2 D)(t_{k+1}-t_k)),\end{aligned}$$

where

$$\begin{aligned}b_{n,k} &= \frac{-2}{\rho_0 z_n' J_2^2(z_n')}(1-r)\int_0^{\rho_0}\rho\Psi_\rho(t_k^-,\rho)\chi_{[0,\rho_{00}]}J_1(\frac{z_n'}{\rho_0}\rho)d\rho\\ &= \frac{-2}{\rho_0 z_n' J_2^2(z_n')}(1-r)\int_0^{\rho_{00}}\rho\Psi_\rho(t_k^-,\rho)J_1(\frac{z_n'}{\rho_0}\rho)d\rho.\end{aligned}$$

In the annulus $B(0,\rho_0)\setminus B(0,\rho_{00})$, the solution is

$$\begin{aligned}\overline{\Psi}(t_{k+1},\rho) = \frac{\Lambda}{\mu_q}&[1-\exp(-\mu_q(t_{k+1}-t_0))]\\ &+[\Psi_0(0)-\sum_{n=1}^{\infty} b_n]\exp(-\mu_q(t_{k+1}-t_0))\\ &+\sum_{n=1}^{\infty} b_n\exp((-\mu_q-\left|\frac{z_n'}{\rho_0}\right|^2 D)(t_{k+1}-t_0)),\end{aligned}$$

where

$$b_n = \frac{-2}{\rho_0 z_n' J_2^2(z_n')}\int_{\rho_{00}}^{\rho_0}\rho(\Psi_0)_\rho(\rho)J_1(\frac{z_n'}{\rho_0}\rho)d\rho.$$

Therefore, we have

$$\Psi(t_{k+1}^-,\rho) = \underline{\Psi}(t_{k+1}^-,\rho)\chi_{[0,\rho_{00}]} + \overline{\Psi}(t_{k+1},\rho)\chi_{(\rho_{00},\rho_0]}.$$

Here, χ is the characteristic function defined by

$$\chi_{[a,b]} = \begin{cases} 1 & \rho\in[a,b] \\ 0 & \text{elsewhere.}\end{cases}$$

Similar to Section 4.1 (fixed spraying applied everywhere), we have

$$\lim_{k\to\infty}\underline{\Psi}(t_{k+1}^-,\rho) \to \frac{\Lambda}{\mu_q}[1-\frac{r\exp(-\mu_q\tau)}{1-(1-r)\exp(-\mu_q\tau)}].$$

The work is identical except for $b_{n,k}$, in which all the integrals are evaluated over $[0,\rho_{00}]$ instead of $[0,\rho_0]$.

Also, it is clear that

$$\lim_{k\to\infty}\overline{\Psi}(t_{k+1},\rho) \to \frac{\Lambda}{\mu_q},$$

because in this case $\exp(-\mu_q(t_{k+1}-t_0)) = \exp(-\mu_q(k+1)\tau) \to 0$. Therefore, as $k\to\infty$, we have

$$\Psi(t_{k+1}^-,\rho) \to \frac{\Lambda}{\mu_q}[1-\frac{r\exp(-\mu_q\tau)}{1-(1-r)\exp(-\mu_q\tau)}]\chi_{[0,\rho_{00}]}(\rho) + \frac{\Lambda}{\mu_q}\chi_{(\rho_{00},\rho_0]}(\rho).$$

Remark 5. Note that, as $k \to \infty$ and $\tau \to 0$, we have

$$\Psi(t_{k+1}^-, \rho) \to 0\chi_{[0,\rho_{00}]} + \frac{\Lambda}{\mu_q}\chi_{(\rho_{00},\rho_0]}.$$

This means that the number of mosquitoes in the interior disc $B(0,\rho_{00})$ approaches zero for sufficiently frequent spraying.

If the number of mosquitoes is less than or equal to a desired value $\hat{\Psi}$, then we have to solve

$$\pi\rho_{00}^2\frac{\Lambda}{\mu_q}[1 - \frac{r\exp(-\mu_q\tau)}{1-(1-r)\exp(-\mu_q\tau)}] + \pi(\rho_0^2 - \rho_{00}^2)\frac{\Lambda}{\mu_q} = \hat{\Psi},$$

or

$$\frac{\Lambda}{\mu_q}[1 - \frac{r\exp(-\mu_q\tau)}{1-(1-r)\exp(-\mu_q\tau)}] = \breve{\Psi},$$

where

$$\begin{aligned}\breve{\Psi} &\equiv \frac{\hat{\Psi}}{\pi\rho_{00}^2} - ((\frac{\rho_0}{\rho_{00}})^2 - 1)\frac{\Lambda}{\mu_q} \\ &= (\frac{\rho_0}{\rho_{00}})^2\tilde{\Psi} - ((\frac{\rho_0}{\rho_{00}})^2 - 1)\frac{\Lambda}{\mu_q}. \qquad (42)\end{aligned}$$

This implies that the minimum spraying effectiveness satisfies

$$\tilde{r} = 1 - [1 - \frac{\Lambda}{\mu_q\breve{\Psi}}(1-\exp(-\mu_q\tau))]\exp(\mu_q\tau),$$

or the minimum spraying period satisfies

$$\tilde{\tau} = -\frac{1}{\mu_q}\ln[\frac{\Lambda - \mu_q\breve{\Psi}}{\Lambda + \mu_q\breve{\Psi}(r-1))}].$$

Therefore, the minimal spraying effectiveness or the minimal spraying period for the fixed spraying in an interior disc is derived, in terms of the birth and death rates of mosquitoes and the spraying effectiveness. This differs from [27] in the term $\breve{\Psi}$ instead of $\tilde{\Psi}$. Note that $\frac{\partial\tau}{\partial\breve{\Psi}} > 0$. It follows that, since the threshold has decreased from $\tilde{\Psi}$ (in [27]) to $\breve{\Psi}$ (see (42)), $\tilde{\tau}$ must be lower ($\breve{\Psi}$ is an increasing function in $(\frac{\rho_0}{\rho_{00}})^2$ and has a maximum equal to $\tilde{\Psi}$ when $\rho_{00} = \rho_0$). Thus, spatial considerations force us to spray more frequently if regular spraying occurs only inside an interior disc.

4.4. **Nonfixed spraying in an interior disc.** Suppose that the assumptions on D still hold. As in Section 4.2 and the previous section, for large m, we can prove that

$$\begin{aligned}\Psi_{m+1}^- \approx \Bigg(&\frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_{m+1}-t_m))] + (1-r)\frac{\Lambda}{\mu_q} \\ &\times [1 - r\exp(-\mu_q(t_m - t_{m-1})]\exp(-\mu_q(t_{m+1}-t_m))\Bigg)\chi_{[0,\rho_{00}]} + \frac{\Lambda}{\mu_q}\chi_{(\rho_{00},\rho_0]}.\end{aligned}$$

Again, define

$$\begin{aligned}\hat{\Psi} \equiv \pi\rho_{00}^2 \Bigg(&\frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_{m+1}-t_m))] \\ &+ (1-r)\frac{\Lambda}{\mu_q}[1-r\exp(-\mu_q(t_m-t_{m-1})]\exp(-\mu_q(t_{m+1}-t_m))\Bigg) \\ &+ \pi(\rho_0^2-\rho_{00}^2)\frac{\Lambda}{\mu_q}. \end{aligned} \tag{43}$$

Now, we can solve (43) to conclude

$$t_{m+1} = t_m - \frac{1}{\mu_q}\ln\Big[\frac{2-r-\frac{\breve{\Psi}\mu_q}{\Lambda}}{1+r(1-r)\exp(-\mu_q(t_m-t_{m-1}))}\Big].$$

Thus, the "next best" spraying events for nonfixed spraying in an interior disc is derived, by assuming that the time between the current spraying and two spraying events previously is sufficiently large. In this case, $\frac{\partial t_{m+1}}{\partial \breve{\Psi}} > 0$ and thus reducing the threshold from $\tilde{\Psi}$ (in [27]) to $\breve{\Psi}$ reduces t_{m+1}. It follows that nonfixed spraying must be initiated earlier if spraying occurs only inside an interior disc.

5. **Wind impact.**

Thus far, we have employed radial symmetry to derive solutions. We now consider the effects of asymmetry on the outcome.

Both infected and noninfected classes of mosquitoes are not only under diffusion due to flying, but may also be under advection by wind. To show the impact of wind, we will add the terms $v \cdot \nabla M$ and $v \cdot \nabla N$ respectively, to the equations (4) and (5) where $v \equiv (v_1, v_2)$ is the wind velocity vector in the polar coordinate systems (ρ, θ) and $\nabla = (\partial_\rho, \frac{1}{\rho}\partial_\theta)$. Therefore, the equations in the ball $B(0, \rho_0)$ are

$$\begin{aligned} M_t &= \Lambda - \mu_q M + v \cdot \nabla M - \beta_m M I + D\Delta M && t \neq t_k \\ N_t &= -\mu_q N + v \cdot \nabla N + \beta_m M I + D\Delta N && t \neq t_k, \end{aligned}$$

with the boundary condition (6) and the impulsive conditions (7)-(8).

Again setting $\Psi = M + N$, in the disc $B(0, \rho_0)$, we have

$$\Psi_t = \Lambda - \mu_q \Psi + v \cdot \nabla \Psi + D\Delta\Psi \qquad t \neq t_k,$$

with the boundary condition (10) and the initial condition (11).

To solve this problem, we define $w(t, \rho, \theta) = \Psi(t, \rho - v_1 t, \theta - v_2 t)$. Then $w_\rho = \Psi_\rho$, $w_\theta = \Psi_\theta$ and $w_t = \Psi_t - v \cdot \nabla\Psi$. Therefore, w satisfies

$$w_t = \Lambda - \mu_q w + D\Delta w \quad \text{in } B(0, \rho_0),$$

which is equipped with the boundary condition

$$w_\rho(t, \rho_0, \theta) = 0 \quad \text{in } \partial B(0, \rho_0) \tag{44}$$

and the impulsive condition

$$w^+ = (1-r)w^-. \tag{45}$$

Note that we have to search for asymmetric solutions because the effect of wind is not radial.

As we did for the radially symmetric solution, we will look at $w = w_s + w_d$, the sum of the steady state and time-dependent solutions.

5.1. **Wind impact steady-state solution.** As in Subsection 3.1, w_s satisfies

$$\Lambda - \mu_q w_s + D\Delta w_s = 0 \quad \text{in } B(0,\rho_0), \tag{46}$$

which is equipped with the boundary condition

$$(w_s)_\rho(\rho_0,\theta) = 0 \quad \text{in } \partial B(0,\rho_0). \tag{47}$$

Consider the eigenvalue problem

$$\Delta\phi(\rho,\theta) = -k\phi(\rho,\theta) \tag{48}$$

$$\phi_\rho(\rho_0,\theta) = 0. \tag{49}$$

with $k \geq 0$ (constant), $0 < \rho < \rho_0$ and $0 < \theta < 2\pi$. This problem can be solved using separation of variables (see [9], page 231). Then we have

$$\theta'' + m^2\theta = 0 \tag{50}$$

$$\rho^2 R'' + \rho R' + (k\rho^2 - m^2)R = 0 \tag{51}$$

$$R'(\rho_0) = 0.$$

Now, (50) has two independent solutions, $\cos m\theta$ and $\sin m\theta$, for $m = 0, 1, 2, \ldots$ Also, (51) has only one regular solution, $J_m(\lambda_{n,m}\rho)$. But $R'(\rho_0) = 0$, which implies $\lambda_{n,m} J'_m(\lambda_{n,m}\rho) = 0$, so we have a nonzero solution only if $\lambda_{n,m} = \frac{z'_{n,m}}{\rho_0}$, where $z'_{n,m}$, $n = 1, 2, \ldots$, is an increasing sequence of all nonnegative roots of J'_m, for $m = 0, 1, 2, \ldots$

As a result, the eigenvector for (48)-(49) corresponding to the eigenvalue $k = \lambda^2_{n,m} = (\frac{z'_{n,m}}{\rho_0})^2$ is

$$\phi(\rho,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\frac{z'_{n,m}}{\rho_0}\rho)(a_{n,m}\cos m\theta + b_{n,m}\sin m\theta). \tag{52}$$

For (46)-(47), we shall try a solution of the form (52). Then we have

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \left(D(\frac{z'_{n,m}}{\rho_0})^2 + \mu_q\right) J_m(\frac{z'_{n,m}}{\rho_0}\rho)(a_{n,m}\cos m\theta + b_{n,m}\sin m\theta) = \Lambda. \tag{53}$$

Note that if we multiply both sides of (53) by $\sin m\theta$, then integrate both sides over $[0, 2\pi]$, we get $b_{n,m} = 0$. Also, if we multiply by $\cos m\theta$, then integrate over $[0, 2\pi]$, we get $a_{n,m} = 0$ for all $m \neq 0$. Therefore, (53) becomes

$$\sum_{n=1}^{\infty} \left(D(\frac{z'_{n,0}}{\rho_0})^2 + \mu_q\right) J_0(\frac{z'_{n,0}}{\rho_0}\rho)a_{n,0} = \Lambda. \tag{54}$$

Now $a_{n,0} = 0$, when $n \neq 1$, which can be proved by differentiating both sides of (54), multiplying both sides of the result by $J'_1(\frac{z'_{n,0}}{\rho_0}\rho)$, then integrating over $[0, \rho_0]$. Then the result follows from (31). Therefore, the only nonzero term is the term with the coefficient $a_{1,0}$. But $z'_{0,1} = 0$ and $J_0(0) = 1$, so $a_{0,1} = \frac{\Lambda}{\mu_q}$.

As a result, we have

$$w_s = \frac{\Lambda}{\mu_q}. \tag{55}$$

5.2. **Wind impact time-dependent solution.** The time-dependent solution w_d is the solution for

$$(w_d)_t = D\Delta w_d - \mu_q w_d,$$

with the boundary condition (44) and the initial condition (45).

To find w_d, we will use the separation of variables $w_d(t,\rho,\theta) = T(t)R(\rho)\Theta(\theta)$. Then we have

$$T' + (\mu_q + kD)T = 0 \tag{56}$$

$$\rho^2 R'' + \rho R' + (\rho^2 k - l)R = 0 \tag{57}$$

$$R'(\rho_0) = 0 \tag{58}$$

$$\Theta'' + l\Theta = 0. \tag{59}$$

Here, k and l are constants.

To have a periodic solution for (59), we should have $l = m^2$ for some $m \in \mathbb{R}$, and this implies that (59) has the two independent solutions, $\cos m\theta$ and $\sin m\theta$. For (57) and (58), we have only one regular solution, $J_m(\frac{z'_{n,m}}{\rho_0}\rho)$, where $z'_{n,m}$, $n = 0,1,2,3,\dots$, is an increasing sequence of all nonnegative zeros of J'_m. Note that this implies that $k = (\frac{z'_{n,m}}{\rho_0})^2$. Finally, (56) has the solution $T(t) = T_0\exp((-\mu_q - (\frac{z'_{n,m}}{\rho_0})^2 D)t)$. As a result,

$$\begin{aligned} w_d(t,\rho,\theta) = a_{0,0}\exp(-\mu_q t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \exp((-\mu_q - (\frac{z'_{n,m}}{\rho_0})^2)t) \\ \times J_m(\frac{z'_{n,m}}{\rho_0}\rho)(a_{n,m}\cos m\theta + b_{n,m}\sin m\theta). \end{aligned} \tag{60}$$

The term $a_{0,0}\exp(-\mu_q t)$ is added because we have to consider the case $k = 0$, at which $J_0(0) = 1$ and $J_m(0) = 0$ for $m > 0$.

5.3. **Wind impact general solution.** Using (55) and (60), we have

$$\begin{aligned} w(t,\rho,\theta) = \frac{\Lambda}{\mu_q} + a_{0,0}\exp(-\mu_q t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \exp((-\mu_q - (\frac{z'_{n,m}}{\rho_0})^2 D)t) \\ \times J_m(\frac{z'_{n,m}}{\rho_0}\rho)(a_{n,m}\cos m\theta + b_{n,m}\sin m\theta). \end{aligned}$$

Here, $a_{n,m}$ and $b_{n,m}$ can be determined using the identities (see [24])

$$\int_0^{2\pi}\int_0^{\rho_0} \rho J_m(\frac{z'_{n,m}}{\rho_0}\rho)J_s(\frac{z'_{s,k}}{\rho_0}\rho)\cos m\theta \sin s\theta d\rho d\theta = 0$$

$$\int_0^{2\pi}\int_0^{\rho_0} \rho J_m(\frac{z'_{n,m}}{\rho_0}\rho)J_s(\frac{z'_{s,k}}{\rho_0}\rho)\cos^2 m\theta d\rho d\theta = \pi\frac{\rho_0^2}{2}(1-(\frac{m}{z'_{n,m}})^2)J_m^2(z'_{n,m})$$

$$\int_0^{2\pi}\int_0^{\rho_0} \rho J_m(\frac{z'_{n,m}}{\rho_0}\rho)J_s(\frac{z'_{s,k}}{\rho_0}\rho)\sin^2 m\theta d\rho d\theta = \pi\frac{\rho_0^2}{2}(1-(\frac{m}{z'_{n,m}})^2)J_m^2(z'_{n,m}).$$

Finally, the total number of mosquitoes is

$$\Psi(t,\rho,\theta) = w(t,\rho + v_1 t,\theta + v_2 t),$$

which is only a shifting for the solution w. In other words, we have

$$\Psi(t,\rho,\theta)=\frac{\Lambda}{\mu_q}+a_{0,0}\exp(-\mu_q t)+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\exp((-\mu_q-(\frac{z'_{n,m}}{\rho_0})^2D)t)$$
$$\times J_m(\frac{z'_{n,m}}{\rho_0}(\rho+v_1t))(a_{n,m}\cos m(\theta+v_2t)+b_{n,m}\sin m(\theta+v_2t)).$$

Remark 6. Note that shifting does not affect the mosquito population, so one can analyse the impulsive system for w instead of Ψ.

To reduce the mosquito population in an interior area $B(0,\rho_{00})$, we have to spray in $B(-vt,\rho_{00})$, because of the wind impact.

A similar analysis to that in Subsection 4.1 implies that

$$\begin{aligned}w(t_{k+1},\rho,\theta)=&\frac{\Lambda}{\mu_q}[1-\exp(-\mu_q(t_{k+1}-t_k))]\\&+(1-r)w(t_k^-,0,\theta)\exp(-\mu_q(t_{k+1}-t_k))\\&+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\exp((-\mu_q-(\frac{z'_{n,m}}{\rho_0})^2D)(t_{k+1}-t_k))\\&\times J_m(\frac{z'_{n,m}}{\rho_0}\rho)(a_{n,m,k}\cos m\theta+b_{n,m,k}\sin m\theta),\end{aligned}\tag{61}$$

where, for $m,n>1$,

$$a_{n,m,k}=\frac{2(1-r)}{\pi\rho_0^2(1-(\frac{m}{z'_{n,m}})^2)J_m^2(z'_{n,m})}\int_0^{2\pi}\int_0^{\rho_0}\rho J_m(\frac{z'_{n,m}}{\rho_0}\rho)w(t_k^-,\rho,\theta)\cos m\theta d\rho d\theta\tag{62}$$

and

$$b_{n,m,k}=\frac{2(1-r)}{\pi\rho_0^2(1-(\frac{m}{z'_{n,m}})^2)J_m^2(z'_{n,m})}\int_0^{2\pi}\int_0^{\rho_0}\rho J_m(\frac{z'_{n,m}}{\rho_0}\rho)w(t_k^-,\rho,\theta)\sin m\theta d\rho d\theta.\tag{63}$$

As a result, we have a recurrence relation for the total number of mosquitoes at any point (ρ,θ), immediately before spraying (equations (61)-(63)). This relation depends on the birth and death rates of mosquitoes, the spraying times, and the spraying effectiveness.

6. Numerical simulations.

In this section, we will estimate the solution for the initial boundary value problem (4)-(8). The data used to find the numerical solution is given in Table 1. D, v_1 and v_2 are estimated by taking the components of the one-dimensional diffusion and the wind velocity in [22] (the angle of projection is 45°), which means dividing the values in [22] by $\sqrt{2}$. The rest of the parameters are taken from [27]. The initial value of mosquitoes is taken as

$$\Psi_0(\rho)=\begin{cases}150 & \rho\in[0,1]\\ 50 & \rho\in(1,3].\end{cases}$$

Note that Ψ_0 is defined as a step function with more mosquitoes in the center region, which represents the area where people and animals live. We assume mosquitoes prefer to live in such areas, where food and water exists.

Parameter	Value	Units	Reference
D	8.838×10^{-3}	km^2 day^{-1}	[22]
Λ	1000	mosquitoes year^{-1}	[23]
μ_q	7.3^{-1}	day^{-1}	[11]
r	$0.92, 0.98$	(proportion)	[27]
ρ_0	3	km	Assumed
ρ_{00}	1	km	Assumed
τ	7	days	Assumed
v_1	-3.5×10^{-2}	km day^{-1}	[22]
v_2	-3.5×10^{-2}	km day^{-1}	[22]
β_m	0.05	mosquitoes^{-1} day^{-1}	[27]
β_h	0.5	humans^{-1} day^{-1}	[27]
h	$1/9$	day^{-1}	[27]
α	$1/8$	day^{-1}	[27]
γ	0.05	day^{-1}	[27]
δ	$1/30$	day^{-1}	[27]
μ_h	$1/50$	year^{-1}	Assumed
π	1	humans day^{-1}	Assumed

TABLE 1. Parameters and sample values.

Due to the cost and logistics of spraying, we will assume that we are interested in either spraying in the central area $B(0, \rho_{00})$, or in the annulus $B(0, \rho_0) \setminus B(0, \rho_{00})$.

From Figure 3, we can see that the number of mosquitoes immediately before spraying is always around 19, while the steady state solution in the absence of spraying is $\frac{\Lambda}{\mu_q} = 20$. However, the mean remains significantly below that of the steady-state solution.

From Figure 4, we can see that the number of mosquitoes immediately before spraying is always around 13, while the steady state solution in the absence of spraying is $\frac{\Lambda}{\mu_q} = 20$. Note that we start with a large number of mosquitoes initially, as in [27].

Figure 5 demonstrates the effect of wind on the mosquito population (without spraying). Instead of remaining in an interior region, wind effects redistribute the mosquitoes (in this case into the back corner).

From Figure 6, we can see contribution of the wind in reducing the effectiveness of spraying, since wind advects mosquitoes away from the area where the spraying is applied. As a result, more mosquitoes are missed by the spraying.

Finally, Figure 7 compares the number of infected humans (solid curve) with the maximal amount of malaria (stars) when spraying is applied every week in the disc $0 \leq \rho \leq 1$. Note that the average number of malaria cases will be lower, since the stars represent the number of infected humans immediately before spraying is applied.

7. Discussion.

The spatial distribution of humans in relation to their environment plays a crucial role in disease management. Understanding the effects of mosquito movements and their relationship with both urban and geographic features is critical in controlling malaria. IRS is a powerful method of malaria control, but it is limited to the physical location of households.

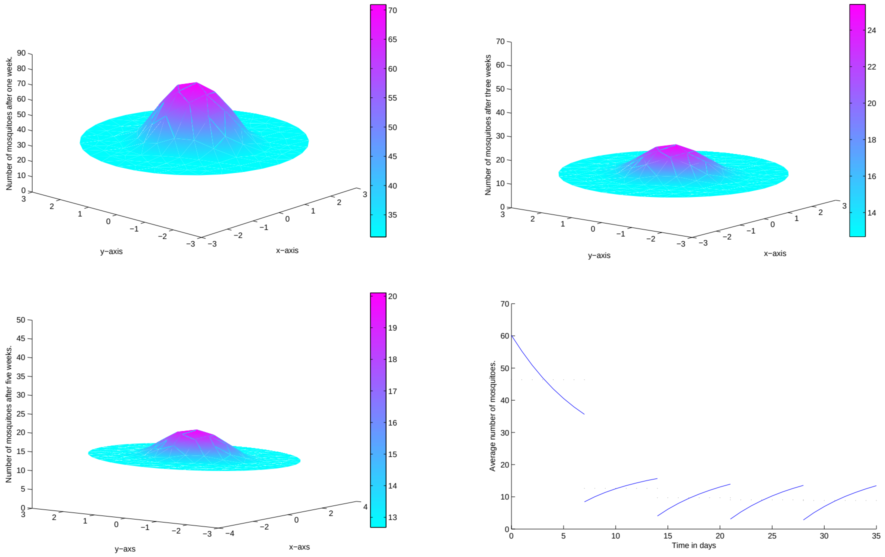

FIGURE 3. Spraying applied every week in the annulus $1 < \rho \leq 3$. Number of mosquitoes after one week (top left), three weeks (top right) and five weeks (bottom left). The average number of mosquitoes in the whole area changes with time (bottom right). The dotted lines illustrate the mean number of mosquitoes over each cycle.

We have used classical methods to solve the nonimpulsive PDEs, then applied impulsive conditions and examined the case of constant initial conditions. If spraying is applied everywhere, we have shown that the results are in line with those from our nonspatial model [27]. If spraying is applied in an interior disc, we employ symmetry to show that results generalise the nonspatial model. In this case, the result of considering such a heterogenous landscape is that spraying has to be applied more frequently, whether it is fixed or nonfixed. Finally, we considered the asymmetric effects of advection due to wind. In this case, we derived a solution to the impulsive system and demonstrated that the results reduce to the previous case, but with a shift.

If the spraying is uniform, then diffusion will not affect the efficiency of spraying. However, if the spraying is applied in part of the area, then diffusion will play a major role and will affect the efficiency of spraying. For example, even if perfect spraying is applied in a central region, mosquitoes will diffuse from the external area and fill the area again. The only way to control mosquitoes is to spray faster than diffusion occurs or to increase the spraying area to take diffusion from nearby areas into account.

There are several limitations to our model, the most obvious being that the only form of symmetry we considered was circular. In particular, real-world heterogenity is unlikely to be circular, as it often involves physical phenomena such as forests, mountains, bodies of water and so on. We also assume that spraying occurs instantaneously, although approximating such events by impulsive differential equations has been shown to be a reasonable approximation, even for quite large delays [28].

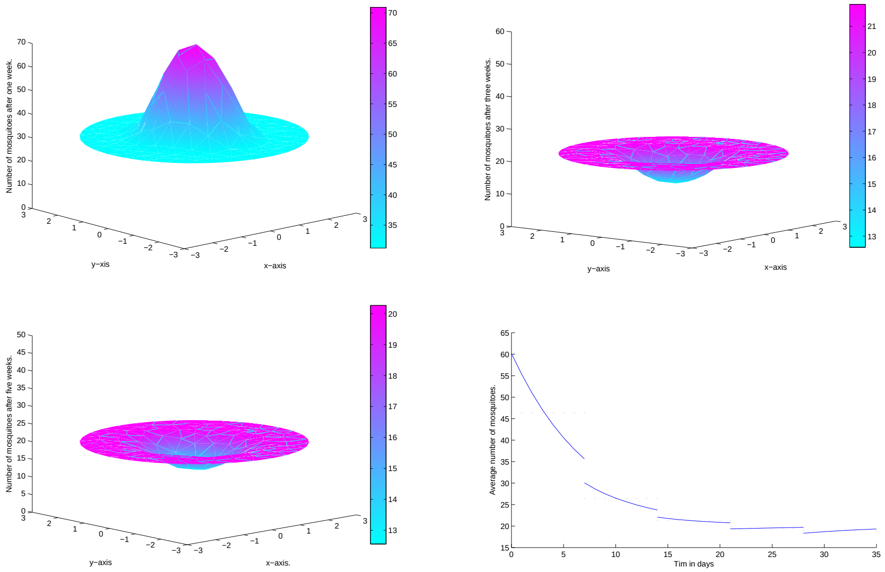

FIGURE 4. Spraying applied every week in the disc $0 \leq \rho \leq 1$. Number of mosquitoes after one week (top left), three weeks (top right) and five weeks (bottom left). The average number of mosquitoes in the whole area changes with time (bottom right). The dotted lines illustrate the mean number of mosquitoes over each cycle. Note that we start with a large number of infected mosquitoes initially, much higher than the steady-state solution.

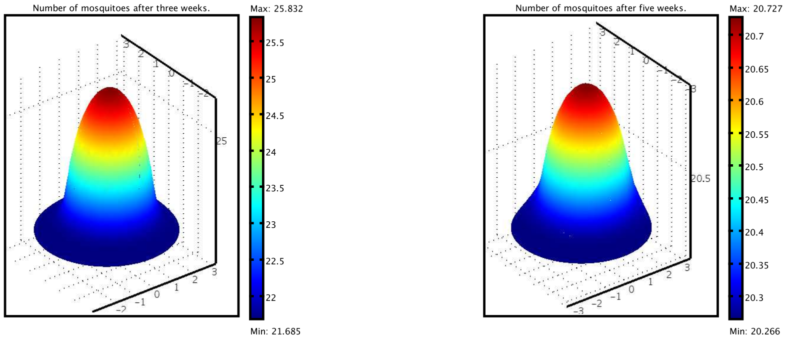

FIGURE 5. The effects of wind in the absence of spraying. Number of mosquitoes after three weeks (left) and five weeks (right). The symmetry is lost, as the wind blows mosquitoes into the back corner.

Furthermore, when considering asymmetrical spread, we ignored the effect of wind upon the spray itself.

Finally, it should be noted that we considered the diffusion coefficients for susceptible and infected mosquitoes to be equal, but this might not hold in general. For example, infected mosquitoes might diffuse slower than susceptibles, at a rate

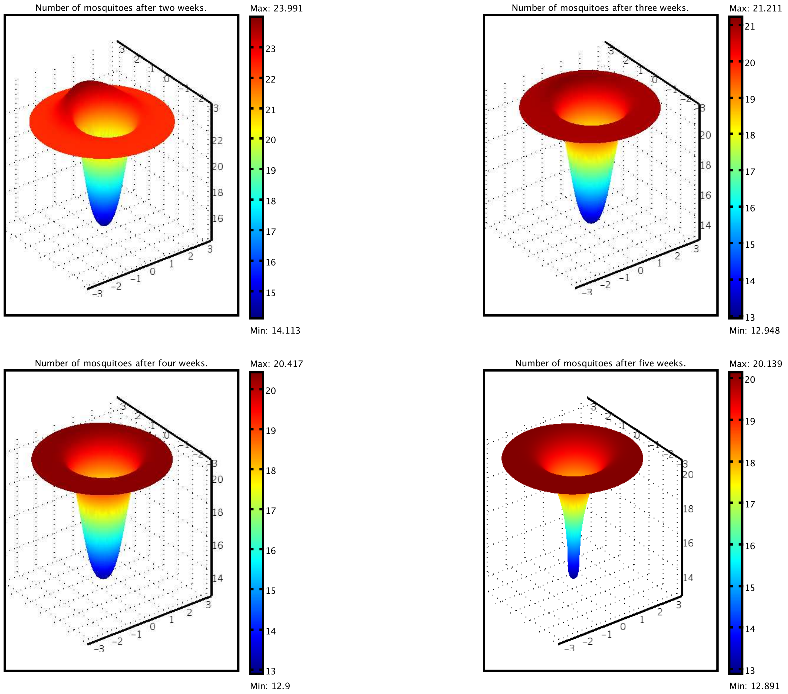

Figure 6. Spraying is applied every week in the disc $0 \leq \rho \leq 1$, with the impact of wind. Number of mosquitoes after two weeks (top left), three weeks (top right), four weeks (bottom left), and five weeks (bottom right). In this case, wind blows mosquitoes outside of the spraying region, reducing our ability to control malaria.

$D_I = D_S - \epsilon$. Numerical simulations based on this formulation showed no appreciable difference (results not shown). Furthermore, the diffusion coefficient may depend on location $D(x, y)$, or on the solution itself $D(M)$ or $D(N)$ or both. While beyond the scope of our work here, such an interpretation would be of interest in the future.

Indoor residual spraying has been extraordinarily effective in global malaria eradication programs in the past, responsible for the eradication of malaria in the developed world and much of the developing world. Although its use today is significantly reduced, and may occur at nonfixed times or in heterogeneous locations, our results demonstrate that these are not obstacles to its effectiveness.

To the best of our knowledge, this is the first PDE model of malaria. Extending the ODE model shows that uniform spraying generalises the model, but incorporating spatial phenomena, even if idealised, results in a variety of possible outcomes, both in symmetric and asymmetric cases. However, control of malaria through IRS is still possible. We thus recommend that indoor residual spraying be reevaluated for widespread control of malaria, in rural as well as urban areas.

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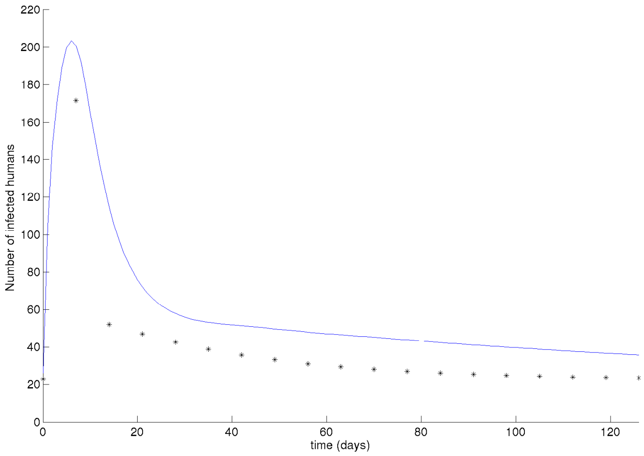

FIGURE 7. Number of infected humans with no spraying (solid curve) and the maximal number of infected humans when spraying is applied every week in the disc $0 \leq \rho \leq 1$ (stars). Note that the latter is an upper bound on the number of malaria cases, since the stars represent the number of infected humans immediately before spraying is applied.

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E-mail address: `malar018@uottawa.ca`
E-mail address: `rsmith43@uottawa.ca`